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Developing mathematical knowledge for teaching in a methods course: the case of function

Michael D. Steele · Amy F. Hillen · Margaret S. Smith

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Abstract This study describes teacher learning in a teaching experiment consisting of a content-focused methods course involving the mathematical knowledge for teaching function. Prospective and practicing teachers in the course showed growth in their ability to define function, to provide examples of functions and link them to the definition, in the connections they could make between function representations, and to consider the role of definition in mathematics and the K-12 classroom. Written assessments, interview data, and class discourse analyses illustrate how the course supported the development of mathematical knowledge that built on individual teachers' prior knowledge as well as the development of a stronger collective understanding of function.

Keywords Teacher knowledge · Teacher preparation · Professional development · Mathematical knowledge for teaching · Functions

Introduction

High-quality mathematics teaching requires deep knowledge of mathematics content, including knowing both how to solve problems and how to represent content in ways that are useful for teaching. These two components of the larger body of *mathematical*

M. D. Steele (✉)
University of Wisconsin-Milwaukee, 398 Enderis Hall, 2400 E. Hartford Ave., Milwaukee, WI 53201, USA
e-mail: mdsteele@uwm.edu

A. F. Hillen
Kennesaw State University, MS 239, Kennesaw, GA 30144, USA
e-mail: ahillen@kennesaw.edu

M. S. Smith
University of Pittsburgh, 5515 Wesley W. Posvar Hall, 230 S Bouquet St., Pittsburgh, PA 15260, USA
e-mail: pegs@pitt.edu

knowledge for teaching (Ball et al. 2008)—common and specialized content knowledge—rest on the premise that teachers are a special class of mathematics users and that what they need to know in order to teach mathematics effectively goes beyond what is needed by other professional users of mathematics. There is substantial evidence that differences in mathematical knowledge for teaching are linked both to the quality of mathematics instruction that teachers enact and the substance of student learning that results (Baumert et al. 2010; Charalambous 2010; Hill et al. 2005; Tchoshanov 2011). There are few studies, however, that explore the ways in which teacher education experiences might provide opportunities for teachers to learn mathematical knowledge for teaching. In this study, we describe changes in secondary mathematics teachers' mathematical knowledge for teaching function through their engagement in a mathematics methods course teaching experiment.

Theoretical perspectives on the development of mathematical knowledge for teaching

Building on Shulman's (1986) call to attend to the intersections between pedagogical and subject-matter knowledge, Ball and colleagues describe six aspects of mathematical knowledge for teaching (MKT), shown in Fig. 1. Subject-matter knowledge, on the left side of the figure, is of particular interest in the context of secondary mathematics teachers because it is presumed to occur through engagement in mathematics content courses that serve diverse learners beyond teachers. It is likely that these courses serve to develop *common content knowledge* (CCK), the ability to use mathematics to solve problems. This knowledge is certainly prerequisite to being able to teach mathematics.

Beyond CCK, teachers also need to know how to represent mathematics content in ways that support student learning of mathematical ideas. Studies have demonstrated the importance of this *specialized content knowledge*[1] (SCK) in teachers' abilities to plan, teach, and reflect on practice, such as anticipating student thinking (Carpenter et al. 1989), identifying and addressing mathematical misconceptions or confusions (e.g., Stein et al. 1990), and making decisions while teaching (e.g., Sherin 2002). SCK is a measurably distinct construct associated with improved student learning outcomes (Brown et al. 2006; Hill et al. 2005; Thompson et al. 2006).

Opportunities for secondary teachers to develop their CCK and SCK related to school mathematics are highly variable, both in the United States and internationally. Typical secondary teacher education programs in the U.S. provide opportunities to learn common content knowledge through tertiary mathematics coursework (Tatto et al. 2012), leaving few opportunities for teachers to develop specialized content knowledge related to the K-12 curriculum and still fewer opportunities to link the two. Ball et al. (2005) called specifically for coursework designed to address SCK, but it is not clear how such coursework might fit into contemporary models of teacher education. An international survey of mathematics teacher preparation programs demonstrates that secondary methods courses are rarely conceptualized as systematic opportunities to learn mathematics content; rather, they focus on planning for and enacting mathematics instruction, fluency with mathematics standards and curriculum, and the development more broadly of mathematical thinking in students (Tatto et al. 2012). While specialized mathematics-for-teachers courses have the potential to develop this knowledge (McCrory et al. 2009), few research-based models exist for such courses in secondary mathematics teacher preparation (for

[1] A third aspect of subject-matter knowledge, Knowledge of Mathematics at the Horizon or Horizon Knowledge, was not clearly defined at the time of the study and was not explicitly included in our design.

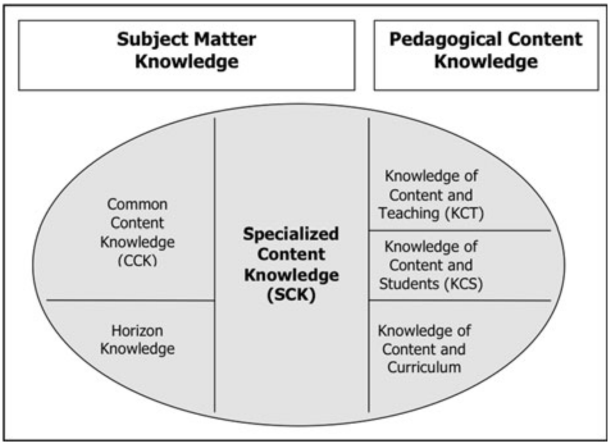

Fig. 1 Aspects of mathematical knowledge for teaching (Ball et al. 2008)

exceptions, see Hill 2006; Senk et al. 2000). There are few models for professional development that feature a sustained focus on the development of mathematics content, and particularly SCK, over time (see Driscoll 1999; Seago et al. 2004 as notable exceptions). Moreover, professional development for practicing teachers tends to be content-general and episodic, despite the existence of content-focused resources (Banilower et al. 2013; Smith 2001). These conditions leave the field with few research-based insights into the opportunities secondary teachers have to learn common and specialized content knowledge related to the mathematics that they will teach to students.

One model integrating CCK and SCK into an existing methods course infrastructure is what Markovitz and Smith (2008) term a *content-focused methods course*. Content-focused methods courses systematically develop a slice of mathematical content within the context of the typical activities in a methods course and are designed to enhance teachers' mathematical knowledge for teaching (Markovitz and Smith 2008; Steele and Hillen 2012). Rather than dealing with mathematics content through isolated tasks, lesson plans, or units, a single content focus spans all work in the course. Common and specialized content knowledge are built in the same systematic ways in which methods courses build teachers' pedagogical knowledge over time.

A content-focused methods course can provide teachers with diverse opportunities to learn mathematical knowledge for teaching. Specifically, we hypothesize that engaging teachers in solving and discussing mathematical tasks from secondary mathematics and then analyzing those tasks from the perspective of both a learner and a teacher support the development of CCK and SCK. This hypothesis rests on three fundamental assumptions. First, solving rich mathematical tasks provides opportunities for teachers to build CCK by doing mathematics. Second, discussing different solution paths with other teachers provides opportunities to develop SCK, both through exposure to other mathematical methods and the work of explaining approaches to one another. Third, considering the same task from the perspective of a learner and a teacher (including the analysis of narrative and written cases of the teaching of that mathematics) provides opportunities to connect teachers' common and specialized content knowledge (Steele 2008).

In the section that follows, we describe how we operationalized CCK and SCK related to the content-focused methods course on algebra as the study of patterns and functions. We present a framework describing aspects of mathematical knowledge for teaching function that framed the teaching experiment and use the framework to analyze teacher learning in the course.

Common and specialized content knowledge for teaching functions

Function is a rich site for studying teacher learning for two reasons. Over the past quarter century, function has emerged as a unifying theme in United States mathematics curricula, shifting away from reserving the concept for upper levels of secondary mathematics. Typically, the definition of function and a connection to graphing occur in the late middle grades and formal study of function in high school, with an emphasis on symbolic and graphical forms and building from simple linear functions to quadratic ones. Study of higher-order, exponential, trigonometric, and parametric functions was historically reserved for upper-division courses targeting college-bound students. This sequence can lead to compartmentalization and fracturing of the function concept (e.g., Vinner and Dreyfus 1989). These challenges, coupled with the fact that the conceptual underpinnings of function are accessible and teachable as early as the intermediate grades (Leinhardt and Steele 2005; Leinhardt et al. 1990), have shifted function's place in school mathematics (NCTM 2000; CCSS 2010).

In a study of teachers' opportunities to learn school mathematics in teacher preparation, the Teacher Education Development Study—Mathematics found highly variable results with respect to the topic of functions. Assessments of US secondary teacher candidates' mathematics content knowledge showed that they performed at or below the mean (Tatto et al. 2012). This suggests that secondary teachers might benefit from additional opportunities to learn the mathematical knowledge for teaching function. This conclusion begs the question, what specific knowledge might benefit teachers in helping their students develop a richer, more connected concept of function? In identifying aspects of common and specialized content knowledge to include in a teacher learning experience, we draw from studies of teachers' knowledge-in-use related to the teaching of function.

Common content knowledge: defining and identifying functions

Function is an important foundational concept in part because it models a wide range of important, predictable phenomena. As such, any user of mathematics would need to know the features that distinguish functions from other relations. Along with this definitional knowledge, mathematics users also need to be able to identify a situation as a function or non-function. Studies of elementary and secondary teachers note that most can articulate a fairly accurate definition for function. Teachers' examples, prototypes, or images, however, encompass only a sub-class of functions most typical in secondary mathematics; functions such as qualitative graphs, functions without a single closed-form rule, or discontinuous functions are less likely to be considered (Even 1990, 1993, 1998; Even and Tirosh 1995; Norman 1992; Pitts 2003; Sanchez and Llinares 2003; Stein et al. 1990; Wilson 1994). These results are consistent with Vinner and Dreyfus's (1989) distinction between a *concept definition* for function and the *concept image* that teachers might hold. While a teacher's concept definition may indeed reflect the defining features of a function, a more limited concept image, such as one restricted to functions that can be expressed with a formula and have a continuous graph, can lead to confusion in identifying functions and non-functions. Simply put, being able to state a definition does not guarantee an understanding of the important features of that definition.

The characteristics of *univalence* and *arbitrariness* distinguish functions from other mathematical relations (Even 1990; Cooney et al. 2011). *Univalence* refers to the mapping of each element of the domain to exactly one element of the range. This characteristic allows for one-to-one and many-to-one relationships, but excludes one-to-many relations. Univalence is often evaluated graphically using the Vertical Line Test, which entails visualizing or drawing a vertical line passing through the Cartesian graph of a function. If the line touches the graph at no more than one point, the graph represents a functional relationship. *Arbitrariness* specifies that elements of the domain and range need not be numbers and that the rule relating them does not have to be described by a regular expression. For example, a function could take as its domain a list of words and return as its range the first vowel in the word. Another example of arbitrariness is a graph of distance over time created by walking in front of a motion detector. Compared to univalence, arbitrariness is less visible as a criterion in definitions of function. Definitions rarely include explicit language related to arbitrary functions, but rather are written such that arbitrariness is not excluded. As such, any mathematically valid definition of function must include *univalence* and not exclude *arbitrariness*.

Specialized content knowledge: recognizing functions in a variety of forms

To foster a rich conceptual understanding of function with students, a teacher needs to know more than simply the definition of function and the criteria by which one can identify examples and non-examples. SCK represents a transition between learner and teacher, in which a teacher mobilizes a deep knowledge of the content to make decisions about what to teach and how to teach it. This knowledge begins with the definition of function, the first of the essential understandings for teachers noted by Cooney et al. (2011). Any definition that includes univalence and does not rule out arbitrariness can be considered a valid definition of function, although such definitions are likely to use diverse language: mappings, inputs and outputs, relationships between sets, and so on. Teachers likely encounter multiple function definitions—in textbooks, supplementary materials, and in understandings their students bring to their classroom—and must determine whether these definitions are valid and/or equivalent. Moreover, some definitions may be more pedagogically useful than others. For example, a class working on functions as numerical tables might find a definition focused on input and output to be useful, whereas a class exploring domains and ranges of composite functions might benefit from a set-mapping definition.

Another important aspect of SCK is the ability to use multiple representations of functions and to translate between them (Cooney et al. 2011). Functions can be represented in a variety of ways—symbolic equations, tables, graphs, mappings, verbally, and by contextual situations[2] (Lesh et al. 1987; Goldin 2002). Secondary mathematics typically privileges the symbolic form, with graphs seen as a less-precise application of the symbolic, and context used largely to spark student interest (Knuth 2000). Heavy emphasis on symbolic representations and easily graphed functions can lead to over-emphasis of continuous linear and quadratic functions and fewer explorations of more exotic functions without a single closed-form rule (e.g., piecewise functions, set mappings, discrete data sets) or whose graphs cannot be easily represented (e.g., the Dirichlet function,[3] $\sin(1/x)$, non-numeric functions). Different representations can help reveal important behaviors to

[2] We use *context* following Lesh et al. (1987) and Van de Walle et al. (2009) to identify real-world narrative function situations as a representation of functions.

[3] The Dirichlet function takes as an input any number, and returns the same number if the input is a rational number, or 0 if the input is not a rational number. Stated another way, $f(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$.

Table 1 Core aspects of teacher knowledge of function

Common content knowledge	Specialized content knowledge
State a definition of function	Evaluate function definitions and consider their utility for teaching
Create and classify examples and non-examples of function	Be able to create and move flexibly between representations of functions

learners; for example, noticing scaling patterns in a table can reveal distinguishing features of proportional and non-proportional functions (Lamon 2012). Analyzing rate of change using the first derivative and considering relative rates of change in a graph can support deeper understanding of the underlying context in a mathematical modeling situation. To promote strong understandings of function in students, a teacher's knowledge of function would include fluency with a variety of representations and the ability to move flexibly between them.

We used this body of research on the teaching of function and teachers' knowledge-in-action to identify important central themes to explore in the content-focused methods course that is the focus of the study described herein. These themes are shown in Table 1 and organized into CCK and SCK categories. These themes should not be taken as everything one might wish teachers to know about function, but rather a pivotal set of concepts related to understanding function that have been shown to be particularly problematic.

We envisioned the content-focused methods course as an opportunity to learn about these aspects of function, but we did not presume that teachers would enter with no knowledge related to function. We instead anticipated a diverse set of needs related to teachers' mathematical knowledge for teaching functions. Some teachers might have misconceptions or confusions related to CCK, in particular with aspects of the definition or mismatches between the definition and examples and non-examples of functions. Others might move between representations of functions in ways common to the secondary mathematics curriculum (e.g., equation to table to graph), but have less experience making connections in other directions. In the sections that follow, we describe the design of the content-focused methods course teaching experiment, describe the population, and draw on course data to answer the research question: what subject-matter knowledge (CCK and SCK) learning is evident from the work and talk of teachers in a content-focused methods course related to functions (herein referred to as the functions course)?

Method

The functions course that is the focus of this study was a teaching experiment implemented as a graduate course for prospective and practicing teachers at a large urban university in the United States. The course was held in a six-week summer term, and met for 3 h twice a week, and was intended to enhance teachers' knowledge of function and to develop their capacity for enacting meaningful student-centered lessons around function concepts. The course was designed as an advanced mathematics methods course, in that it aimed pedagogically to develop teachers' capacities to identify worthwhile mathematical tasks, plan rich student-centered lessons, and analyze student work and student thinking. In planning the course, mathematical tasks, narrative and video cases, and student work relating to

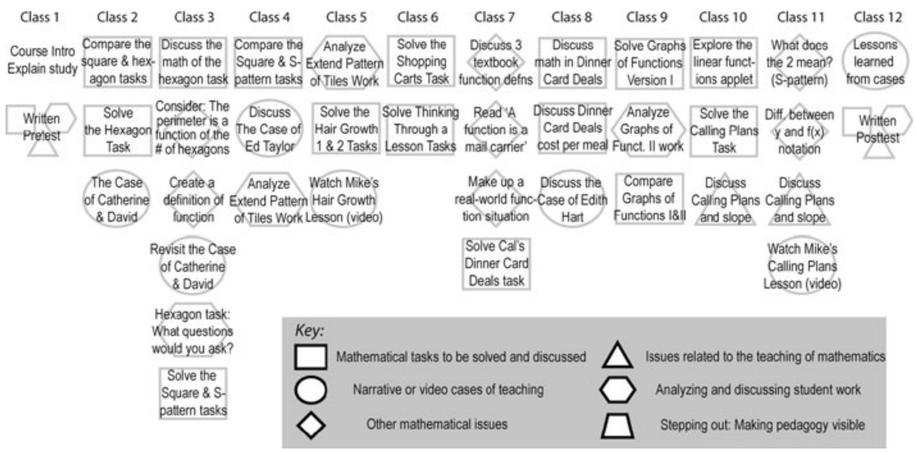

Fig. 2 A course map representing the progression of activities in the course

function were integrated into the methods structure, and a set of mathematical learning goals were developed alongside the pedagogical goals. Mathematically, the course aimed to: (1) develop a mathematically accurate definition of function and use it to distinguish examples and non-examples of function; (2) distinguish types of functions, such as linear/nonlinear and proportional/non-proportional; (3) solve a variety of problems involving functions, exploring a variety of terminology and notation; and (4) recognize the equivalence of and be able to use in practice a variety of function representations. Figure 2 shows the activities included in the course, with different shapes used to denote different types of activity.

The activity sequences shown in Fig. 2 begin with exploring mathematical tasks (rectangles) as starting points to consider issues of teaching and learning mathematics through subsequent activities such as analyzing cases of teaching and student work from the same or a similar task. As such, both the mathematical and pedagogical experiences that followed provided teachers with sites for building content knowledge. The course modeled a type of teaching whose goal was for teachers to learn with understanding (Carpenter and Lehrer 1999) and to connect their experiences as mathematical learners to their work as mathematics teachers. The instructor (also the third author) kept public discussions learner- and content-centered; all teacher contributions and ideas were welcomed, with selected (often controversial) ideas explicitly taken up by the instructor for further discussion. These design features embodied the three fundamental assumptions regarding the development of CCK and SCK: that solving rich mathematical tasks can build CCK, discussing different solution paths can develop SCK, and investigating tasks from the perspective of a learner and a teacher can foster connections between CCK and SCK. The course viewed this knowledge-building as both individual and collective, visible in teachers' written work and talk.

Course design began with the identification of function-related mathematical tasks, cases, and student work that represented a variety of types of functions and non-functions. In drawing on existing narrative and video portrayals of practice, the course was anchored by linear and quadratic relationships most common in secondary mathematics. The research and design team worked to add additional tasks and student work samples from this core and to design other activities that would afford consideration of a wider array of functions, non-functions, and representations. The research and design team, which included the instructor, met weekly to assess teacher learning and make recommendations regarding subsequent content. This allowed the team to make adjustments to the mathematical storyline of the course and thoughtfully select mathematical tasks to push teachers' thinking based on their performance in the course. As such, we take as data both the work and talk produced by teachers and the selection and design of course activities, which were responsive to emerging teacher thinking.

Participants

The 21 teachers in the course varied with respect to certification, subject-matter preparation, and experience. Thirteen were completing a requirement for elementary or secondary certification and a Masters of Arts in Teaching degree (MAT); six were certified practicing teachers enrolled as part of a Masters of Education program, and the remaining two teachers were students in deaf education interested in mathematics. Sixteen teachers were secondary-certified and had earned a bachelors degree in mathematics (or its equivalent), with five elementary- or special education-certified teachers holding a bachelors degree in a discipline other than mathematics.

Data sources

Data sources related to mathematical knowledge for teaching function were drawn from a larger set of instruments designed to capture teacher learning from the course (including learning related to the pedagogical goals). For this study, we identified written tasks and class discussions with the potential to illuminate teacher learning of subject-matter knowledge related to function. We examined the ways in which teachers solved mathematical tasks, discussed different solution paths, and considered the tasks as learners and teachers, selecting data sources that represented both individual and collective work and talk on tasks and the teaching of those tasks. These sources included a pre- and post-course written assessment of key mathematical and pedagogical ideas (during Classes 1 and 12, respectively); two audiotaped interviews with each teacher—the first conducted near the beginning of the course and the second after the last class session; video records of class meetings as well as field notes and instructional artifacts for each meeting; and copies of teachers' work. Data sources used for this analysis are shown in Table 2; all tasks described can be found in the Appendix.

Videos of the class meetings and teacher interviews were transcribed and verified by the research team. Teachers were given a course map similar to Fig. 2 to assist in describing activities that contributed to their learning of mathematics in the post-course interview.

Data analysis

The data analysis followed a constant comparative method that balanced the individual teacher and the collective with respect to learning. The research team moved between individual results from written assessments and interviews and the class discourse and artifacts constituting collective opportunities to learn. Data coding followed a two-stage process, with the first stage focused on the *work* produced by individual teachers. We characterized change in common and specialized content knowledge by analyzing written and interview tasks relevant to the aspects of CCK and SCK in Table 1. Rubrics were developed for written items, with change assessed using parametric and nonparametric

Table 2 Data sources and analysis

a. Subset of data sources in this study

Data source	Description
Pre-/post-test Questions 1 and 3	Presented several steps of visual patterns representing underlying linear functions Teachers asked to find perimeter (Q1) or number of blocks (Q3) for next step, 10th step, generalize for any step
Pre-/post-test Question 2	Described 3 contextual situations, teachers asked to represent each in words, as a table, equation, and graph and to classify situations as function/non-function, linear/nonlinear, proportional/non-proportional
Pre-/post-test Question 4	Asked teachers to define function and provide an example and non-example
Post-course interview: reflection on learning	Teachers described what they learned during the course and identified specific activities on the course map (similar to Fig. 1) that supported their learning
Class discussions	Discussions identified by at least seven participants in post-course interview as a source of their mathematical learning

b. Relationships between data sources and mathematical knowledge for teaching function

Type of knowledge	Data sources	Analysis methods
Common content knowledge: State a definition of function	Pre-/post-test Question 4 Post-course interview	Correct/incorrect/inconclusive (Q4) Descriptions of what teachers learned
Common content knowledge: Create and classify examples and non-examples of function	Pre-/post-test Questions 2 and 4 Task selected for inclusion in course	Correct identification of example (Q2) Correct/incorrect example and non-example (Q4) Function type, starting representation
Specialized content knowledge: Be able to create and move flexibly between representations of functions	Pre-/post-test Questions 1 and 3 Post-course interview Discussion of Cal's Dinner Card Deals	Rubric for representations (see Table 3) Descriptions of what teachers learned Mapping of connections between representations
Specialized content knowledge: Evaluate function definitions and consider their utility for teaching	Post-course interview	Descriptions of what teachers learned

Table 3 Representations rubric for Questions 1 and 3

Score	Description	Example (using Question 1)
4	*Full explanation; well connected to visual pattern* A generalization is evident (verbally or symbolically) All aspects of the generalization are explained accurately with respect to the visual pattern	For each pentagon on the end of the train, you count 4 sides, so that is always $4 \times 2 = 8$. There are two less pentagons in the middle of the train than the train number itself, and each of these has 3 sides counted as part of the perimeter (3 exterior sides)— $>8 + 3(n - 2)$, where n is the train number
3	*Some explanation; partially connected to visual pattern* A generalization is evident (verbally or symbolically) At least one aspect of the generalization is explained accurately with respect to the visual pattern Remaining aspects of the generalization are either explained incorrectly, inaccurately, vaguely, or not explained at all with respect to the visual pattern	$n(5) - (n - 1)(2)$ n refers to the number of the train, multiply this number by 5 then subtract one less than the total number multiplied by 2 From the visual representation, we can see that 2 pentagons will share one side. This shared side will be on the inside of the shape and will not be included in the perimeter. This shared side must be subtracted from each pentagon
2	*Weak explanation; some connection to visual pattern* A generalization is evident (verbally or symbolically) At least one aspect of the generalization is explained, but the explanation is incorrect, inaccurate, or vague	$(3x) + 2$ When a new train is added only three-unit sides, two sides of that train are actually added. The $(3x)$ is 3 sides of the trains from before plus multiplied by the train number
1	*Numeric explanation only; no connection to visual pattern* A generalization is evident (verbally or symbolically) The elements of the generalization are explained but not connected to the visual pattern in any way	$3n + 2$ Multiply the number of trains by 3 and then add 2. I know this works because it fits my pattern. My description is independent of the visual representation. I had to make a table—the pictures did not help me in finding the patterns
0	No explanation present	

tests of significance as appropriate. These items were coded by the first two authors with an inter-rater reliability of at least 92 %. Teachers' responses to interview questions related to their mathematical knowledge for teaching function were also examined and coded thematically. Four written assessment items and one interview question were included in the coding.

The first phase of analysis focused on CCK and SCK as demonstrated on the pre- and post-course assessment. Assessment item 4 was used to measure teachers' CCK related to the definition and ability to identify examples and non-examples. Definitions were coded as correct if they included the idea of univalence and not explicitly rule out arbitrariness. Specifically, teachers' definitions needed to include the idea that y is a function of x when there is a relationship between x and y such that for every value of x there is one and only one corresponding value of y. The terms x and y could be replaced by similar mathematical terms such as input and output, domain and range, and independent variable and dependent variable. Definitions that did not include the idea of univalence or made erroneous statements (e.g., functions must be linear relationships) were coded as incorrect. Definitions were coded as inconclusive if there was not enough information present to suggest that the teacher included the univalence criterion. Such definitions included correct statements (e.g., functions pass the vertical line test) that provided no further explanation regarding why the statements implied a relationship was or was not a function. To describe teachers' ability to classify examples and non-examples of functions, items 2 and 4 were used. In addition to providing a definition, item 4 also asked teachers to provide an example and non-example of a function; these examples were coded as correct, incorrect, or inconclusive and categorized by mathematical representations used. Item 2 asked teachers to identify examples as functions or non-functions; responses were coded as correct and incorrect.

Items 1 and 3 were used to measure teachers' SCK related to creating and connecting representations of functions. Responses to the items were coded using a rubric (see Table 3) that captured the degree to which teachers made connections between representations of functions, and specifically between the visual representation, their written representation of the solution, and any other representations (most frequently tables and symbols) used in the service of solving the task. No item directly measured teachers' ability to evaluate function definitions and their utility for teaching; however, several in-class conversations focused explicitly on this topic and were analyzed to describe teacher learning.

The second phase of the analysis considered the *talk* in which teachers engaged while solving tasks and discussing the teaching of those tasks. We considered talk a significant data source because it represented teachers' abilities to engage in the practices of solving tasks, discussing solution paths, and analyzing tasks as a learner and a teacher, mirroring the work we would anticipate teachers doing as a professional learning community in a school. We identified class conversations to analyze in a two-step process. First, we used responses to an interview item in which teachers were asked to identify activities that contributed to their learning about mathematics. We noted discussions of activities that were identified as sources of mathematical learning for at least seven of the twenty-one teachers. Seven was selected as the threshold as it represented one-third of the course population and guaranteed diversity across constituencies (elementary/secondary, prospective/practicing) in the course. This process yielded eleven discussions in six class meetings (Classes 2, 3, 5, 7, 8, and 11).[4]

[4] Classes 1 and 12 consisted almost entirely of data collection or administrative work; thus, the six classes identified as containing relevant activities represent over half of the course. Given that the course also had goals and activities related to developing pedagogical knowledge, we feel this is a fair representation of the extent to which content was a focus of the methods course.

After identifying these conversations, we performed a content analysis on the transcripts to identify the most significant opportunities to learn the four aspects of CCK and SCK in Table 1 (Babbie 2012). Using the descriptors to identify themes, we wrote analytical memos for each identified conversation relative to any of the four aspects present and identified excerpts that were characteristic of the conversation. We present the results of our analysis of the assessment items, with descriptive detail and transcript excerpts from the discussions, to describe changes in teachers' CCK and SCK related to function.

Results

In this section, we consider the four aspects of function related to CCK and SCK and describe changes in teacher knowledge, as evidenced by the work and talk as teachers solved tasks, shared solutions, and discussed the teaching of tasks related to function. By the close of the course, teachers had changed in the ways in which they defined function, how they identified examples and non-examples of functions, the representations they created and the connections between them, and their conceptions of the role of definition in the teaching and learning of function. Looking across the twelve class meetings and associated activities, over half of the talk in the course related to CCK and SCK of function (rectangles/rhombi in Fig. 2).

Common content knowledge: defining function

Assessment item 4 asked teachers to provide a definition of function. Pre-test definitions indicated that fewer than half of the twenty-one teachers provided a correct definition of function, as shown in Table 4. On the post-test, twenty teachers provided a correct definition of function, a significant difference (Fisher's exact test, $p < 0.001$). All twelve teachers who did not provide a correct definition on the pre-test improved in some way: seven moved from incorrect to correct, four from inconclusive to correct, and one from incorrect to inconclusive. There were also shifts in attention to the criterion of arbitrariness: definitions that explicitly ruled out arbitrariness declined from seven on the pre-test to zero on the post-test. There was a significant increase in the use of *input* and *output* in teachers' definitions[5]; on the pre-test, three teachers used these terms, whereas twelve used them on the post-test (Fisher's exact test, $p < 0.01$).

Reflecting on their learning in the post-course interview, eleven teachers said they learned something about the definition of function. Ten were secondary teachers, the population one might consider most likely to already have robust knowledge related to function, including all the practicing secondary teachers. These eleven were not just the teachers whose written definitions changed: four moved from incorrect to correct on the assessment, three from inconclusive to correct, and four produced a correct definition both times. Carl, a prospective secondary teacher who moved from incorrect to correct, described his learning about the definition of function:

> I have a clearer definition of what a function is. I think most of us came into the class having worked with functions before, obviously… but I don't think a lot of us really

[5] This is not particularly surprising, since these terms were used in the class-created definition that was posted during the course and were frequently used by teachers to make sense of function. However, the fact that there was not a complete convergence to this language, along with the diversity in the construction of input–output definitions, suggests that teachers did not simply memorize the class definition.

Table 4 Teachers' written definitions of function

	Pre-test (number of teachers)	Post-test (number of teachers)
Correct	9*	20*
Vague/inconclusive	4	1
Incorrect	8	0

* $p < 0.001$

What is a function?

1. specifies a relationship between 2 sets – domain and range
2. output <u>depends on</u> the input (perimeter of polygons depends on the number of polygons)
3. for each input there is a unique – EXACTLY ONE – output
4. graph would pass the vertical line test
5. some "things" that pass the vertical line test have the same output for 2 different inputs

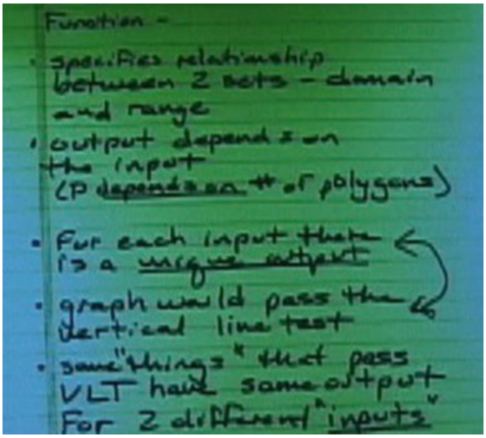

Fig. 3 The final co-constructed definition of function

> had a really SOLID DEFINITION in our heads of what a function is. And I think that the class helped us revise our own inkling of what a function is… Before the course, if someone had asked me, "What is a function?," I would've said something about the vertical line test. I would've said something about an equation, or function notation. But I don't think I could've really given a really direct answer. After the course, I think I can.

Content analysis of talk

The work on definition of function was designed to do more than arrive at an accurate definition. The discussions aimed to build a conceptual understanding of the meaning of the term and the elements of the definition. The term function was publicly introduced by a secondary teacher in Class 2, and explicit consideration of the definition (first constructing a definition, then revisiting and refining the definition) took place in classes 3, 4, and 7. Final proposals for definitions are shown in Fig. 3. It is notable that three of the items on the list (items 3, 4, and 5) address the idea of univalence, while the remaining items (items 1 and 2) help specify the components in a functional relationship.

After the term function emerged in Class 2, the instructor flagged it and indicated that they would return to that statement in the next class. In Class 3, teachers were first asked whether they agreed with the claim that pattern task they solved represented a function (a unanimous yes) and asked groups to answer, “What is a function?” During the public discussion, several aspects of a definition were offered, critiqued by other teachers in the class, discussed, and revised, with the instructor recording all comments. The discussion ended when teachers did not have any further additions to or comments on the list. Perhaps they thought they had accomplished the goal of defining function and were ready to move on, much as a typical instructional progression might unfold. However, the instructor noted that this was a work in progress.

> Well, it’s actually quite a list of things that we’ve got here. What I’m going to do is, produce that [on chart paper] and what we’ll try to do is revisit it and see if we want to modify it, and at some point, what we’re going to want to do is to talk about - do we need all of these pieces in order to have fully defined a function? Which of these components are critical to the definition, and which are just saying the same thing in a different way.

The instructor provided teachers with opportunities, both implicit and explicit, in subsequent classes to revisit the definition and revise it. The list was brought out during activities in which there was the potential for the definition to be revisited and revised. Teachers first grappled with the univalence criteria, as seen in this excerpt debating whether the graph of $y = x^2 + 1$ is a function from Class 4:

Instructor:	So, if you had a horizontal line test, ok, it’s going to hit twice. So why isn’t that a problem for a function? Kelsey?
Kelsey:	I don’t think it’s unique, as so much as it’s exactly one. Like that there can be just ONE. It doesn’t have to be different from any other one. The thing is, for every input, there can be ONLY one output
Instructor:	Ok, so for every input, there can only be one output. So where does uniqueness come into play though? Bruce?
Bruce:	I think when you’re talking about a function, you’re more concerned about the dependency between variables… So I think that’s how I keep it in my mind, what’s a function and what’s not. If I input a value, I want to know what’s going to come out. I’m not really concerned about the other way around, trying to figure out where I started with my output. Because in the real world, there’s usually a dependency between variables, the independent and the dependent variables. So you’re worried about, taking a control variable, and does it actually control the situation. If it does, then, it’s a function
Instructor:	Ok. Bonnie?
Bonnie:	In order for it to control the situation, there’s got to be just one output. You have to be able to determine what that output is
Instructor:	So there’s one output, for an input. So when I put in one, I get two
Bruce:	Right
Instructor:	When I put in one, I don’t get anything else but two
Kelsey:	Right, right
Instructor:	Ok, so that’s the uniqueness we’re talking about. Bonnie?

Bonnie: That's just what I was going to say. The uniqueness, is to that particular input, it's not that it's unique in all of the functions

[…teachers continue to debate the issue and illustrate with examples…]

Bruce: [referring to the chart paper] I still agree with Christopher, though. I mean, I would change that unique to just say one. It's just more clear. I mean, we could explain it but for someone to look at it, if it said "one", it'd be more clear
Kelsey: Exactly one
Several: Exactly one
Bruce: Exactly one
Teacher: What's wrong with unique?
Instructor: Well, part of the reason for doing this list was so that we could modify it as we worked. I didn't cross unique out, but I added exactly one. Ok?

The more challenging criterion, arbitrariness, was addressed several ways. The most salient activity that provided teachers opportunity to learn was discussion of the article, "A Function is a Mail Carrier," in which an analogy of a mail carrier (with letters as inputs and mailboxes as outputs) is used to make sense of function (Sand 1996). This discussion was noted by over half the teachers as contributing to their learning about the definition of function. In an activity that followed in which teachers were asked to create real-world examples of functions, their work showed explicit consideration of arbitrariness. The task asked teachers to create multiple representations of each relationship (table, graph, equation) as appropriate and explain why each was a function. In the following excerpt from Class 7, prospective secondary teacher Bert shares his example:

Bert: I just used students' test scores as an example. I said that when students take tests, that they each have a test score. And they can score the same on a test, but for each student they're not gonna come out with two different scores. So, I came up with this scatter plot of students and their score. So student A had one score, B had a different one, C had a different one, and then I made D have the same as B. So then I drew the mapping of the student and their score so that B and D, although they have the same score, you can see that for each student, they had exactly one score. No student had two different scores. You can only come up with one score. So, the score was a function of the student. The score corresponds to each student. What score you have corresponds to what the student got on the test. I couldn't really come up with an equation for it, so I didn't
Instructor: Is that legitimate, not having an equation?
Several: Mhm
Instructor: Yeah, it would be. Not everything has to be defined by an equation

Looking across these data, we see teachers that had difficulty initially providing a complete and accurate definition of function on the pre-course assessment. The ideas shared in the first discussion, as captured in Fig. 3, suggest that the teachers as a group were able to identify key features of function in definition-like statements. This is not terribly surprising despite the poor results on the pre-test, as most teachers in the class would have had exposure to the definition of function as learners, and some would have taught the concept to their own students. Echoing previous research, arbitrariness was not explicitly considered in teachers' first definitions. The discussion of non-numeric functions in the course led both to teachers making use of arbitrariness in creating new examples as seen in Class 7, and attention to the criterion in their post-course assessment written definitions. This demonstrates that through their work in the course, teachers strengthened their conception of function as a construct and the key criteria that need to be present in a definition of function.

Common content knowledge: examples of functions

The ability to classify mathematical situations as functions or non-functions is the second aspect of CCK measured on the written assessments. Item 4 asked teachers to create an example and a non-example of a function. Item 2 presented three context-based situations and asked teachers to classify the relationships as functions or non-functions. Teachers exhibited strong CCK on both items, even on the pre-test. Table 5 shows the results of teacher-produced examples and non-examples. The data indicate a ceiling effect for the example, with nearly all of the teachers providing a correct example of a function on the pre-test, with all but two correct examples either linear or quadratic functions. There was a significant increase in correct non-examples of function (Fisher's exact test, $p < 0.01$). Data from the three contextual situations also showed a ceiling effect, with the only significant change being an improvement in identifying the rational function in example b as a function ($t(20) = 1.72$, $p = 0.01$).

Given the background of these teachers, it is no surprise that the majority could easily produce and identify examples of functions. The examples provided by teachers were relatively straightforward relationships central to secondary mathematics. However, it is interesting to note that prior to the course, the majority of teachers were able to provide correct examples of functions, but not all teachers were able to correctly produce a definition of function.

Content analysis of talk

During the course, teachers considered a wide range of mathematical situations and classified them as examples and non-examples of function in two ways. The first way was identifying a situation as a function or non-function—a common and expected type of discourse. The second way was more unique—using examples of functional relationships to reconsider and revise the definition of function. These conversations occurred both spontaneously, when teachers used the term *function* and/or the definition while exploring examples, and intentionally, when the instructor specifically asked teachers to relate an example to the definition. Bringing the definition to bear on examples was a challenge for

Table 5 Teachers' examples and non-examples of function

	Example		Non-example	
	Pre-test (number of teachers)	Post-test (number of teachers)	Pre-test (number of teachers)	Post-test (number of teachers)
Correct	19	20	13*	19*
Vague/inconclusive	2	0	2	0
Incorrect	0	0	6	0
Did not respond	0	1	0	2

* $p < 0.01$

teachers, not necessarily because they could not do so but because they were reluctant to tie their examples to specific features of the definition. Teachers initially preferred to simply state that the example met the criteria upon which they had decided. This was evident in this excerpt from Class 4, in which teachers were comparing linear (square pattern) and quadratic (s-pattern) visual pattern tasks that they had solved:

Instructor: So repeat what you said now, Owen
Owen: I was just explaining, there's a domain, because it's a function
Instructor: So, that's something that hasn't been said, they're functions. Are they?
Owen: Mhm
Several: Yeah
Instructor: Yeah? How do you know?
Bert: 'Cause they'd meet the criteria on the list
Several: Criteria on the list
Instructor: They meet the criteria on our list. I just happen to have the list right here. I basically took what we did, on a piece of paper, and-
Several: [laughing] Oh, it's OK. It's OK. We know what it says
Instructor: [laughing] You know what it says. You've committed it to memory. I just simply reproduced it on a larger sheet. Ok, [points to first bullet] specifies relationship between two sets, a domain and range. Do they both do that?
Several: Mhm
Instructor: Ok, so what is the relationship here?
Bert: Multiply your domain by 4, and add 4
Instructor: Multiply by 4, and add 4. So the domain is?

As the discussion continued, the instructor pressed teachers to connect features of the mathematical relationship to particular aspects of their definition, such as domain. Teachers took this practice up in subsequent discussions, spontaneously offering such details when examining other examples in the course.

In other discussions, teachers used examples to highlight particular aspects of what it might mean to be a function. This excerpt illustrates a moment in which the class definition of function was called into question when considering a new example. Teachers were creating a graph of situation in which a girl's hair grows 1 inch each month, and she gets her hair cut by 1 inch every second month. A debate about representing the haircut erupted—do we use an open and a closed circle, two open circles, two closed circles, a slanted line, or something completely different:

Instructor: Ok, so I want to talk about that, wouldn't be a function. So let's take all those circles and fill them in, ok? Part of the reason for bringing this PARTICULAR problem up is to really call into question, what does it really mean to be a function, and if those were filled in, why wouldn't it be. You said the vertical line test, but this is actually an opportunity to try to contextualize what that means, yet again, in an example that's slightly different than ones we've looked at before
Bonnie: You would have two different lengths of hair and every second month… there would be two different lengths of hair in that one month. So for that input there'd be two outputs rather than only one
Bert: I don't want to start a big debate over this, but when you think about it realistically, you could close them both in but they wouldn't be above each

other. They'd have to be spread out…. it wouldn't be at the same time interval. So technically, you could have them both closed

Instructor: Well, time is continuous… so the point here isn't really to try to argue which it should be. It's really just an opportunity to do what Elaine was saying, in trying to think about how you'd have to define what it is you're looking at, what the domain and range of the variables are. And also to start thinking about what it means to not be a function. I'm not arguing that this isn't

This excerpt summarizes how examples compelled teachers to revisit aspects of the definition. There was a strong sense that this relationship should be a function, leading teachers to consider how to either select a representation that would satisfy the definition or modify the definition such that the relationship could be classified as a function.

While classifying examples with respect to a definition is commonplace, using examples to reconsider a definition is a rare mathematical practice. Vinner (1991) notes that most textbooks in secondary and collegiate mathematics position definitions as the means by which one acquires a concept and then uses that definition to solve problems and classify examples. Teachers' adoption of this rare practice was likely to have been a novel opportunity to consider how definitions and examples can interact. These data show that teachers improved in their abilities to classify examples as functions or non-functions and also used the consideration of examples to revisit and refine their definition. This interaction is indicative of opportunities to link two aspects of CCK, defining function and considering examples. This interaction sets the stage for considering the role that various representations of function examples can play in supporting student learning.

Specialized content knowledge: connecting representations of function

As suggested in the excerpts involving examples of functions, the representation a mathematical relationship takes can profoundly influence how one understands the underlying relationship. As such, the course presented mathematical tasks that used a range of starting representations (equation, table, graph, context), with the instructor systematically pressing teachers to produce and connect different representations of function situations. Varying the starting representations afforded teachers the opportunity to move between multiple representations. Table 6 shows the starting representations used in the course tasks.

Representations were a significant source of learning for teachers in the course, as reported in the post-course interview. Twenty of twenty-one teachers offered that they learned about representations, connections between representations, and the importance of using and linking multiple representations in their teaching. Many teachers noted that their representational fluency was limited before the course and made explicit connections between the importance of multiple representations and teaching practice. The following excerpts are representative of the ways teachers described their learning:

Bert: I think now I understand connections, not just the equations not just the graphs and the tables but to the initial diagram. You're seeing where the actual numbers come into play with the visual representation. You can see that your slope is going to be your change in the table or your rise over run on the graph, and your y-intercept is going to be on the graph. The y-intercept will be your starting point on the table whenever x is zero

Table 6 Range of examples used in course tasks

Class	Example (starting representation in parentheses)						
	Functions						Non-functions
	Linear proportional	Linear non-proportional	Quadratic	Piecewise	Rational	Non-numeric	
1							
2		Hexagon task (context)					
3			S-pattern task (context)				
4		Square/Pool border task (context)					
5		*Paul's hair growth (context/table)*		*Sonya's hair growth (context)*			
6		Supermarket carts (context)					
7	Car wash* (context) Cal's Dinner Cards: Regular Plan (graph)	Cal's DinnerCards: Plans A and B (graph)				*Mail carrier (context)* *Students & test scores (graph)*	Weight and height (graph)
8		Cal's Cost Per Meal: Regular Plan (table)			Cal's Cost Per Meal: Plans A and B (table)		
9	*Graphs of functions: Functions 1 & 2 (symbolic)*	*Graphs of functions: Functions 3 & 4 (symbolic)*					
10		Calling Plans (context)					
11							
12			S-pattern task (context)				

Italics indicate situations that are or could be considered continuous; non-italics indicate discrete situations

Diane: [One idea I learned] was multiple representations, being able to see [slope in] the different representations. You can see it in a graph, or you can see it in a table…. I usually would think of [functions] algebraically, but I never thought of trying to look at all representations for the same function, and see the similarities and the differences

Items 1 and 3 on the written assessment measured changes in teachers' representational fluency. Teachers' scores on a rubric (see Table 3) measuring use of and connections between representations show significant improvement. These items asked teachers to create a generalization for two visual patterns and to link aspects of their generalization to the visual. On both items, teachers showed a significant increase in their capacity to connect their generalization to the visual pattern (Wilcoxon sign-rank test; Q1: $W = -73$, $n_{\mathrm{s/r}} = 12$, $p = 0.0045$; Q3: $W = -91$, $n_{\mathrm{s/r}} = 13$, $p = 0.0016$).

Content analysis of talk

Talk about representations of functions was pervasive throughout the course. Early in the course, a version of the representation diagram adapted from Lesh et al. (1987) was introduced as a reference. The diagram as it was presented is shown in Fig. 4a. During subsequent work on mathematical tasks, teachers were pressed to create and share a variety of representations and to connect the key mathematical features of the situation to those representations. As teachers encountered tasks with different starting representations, teachers began to appreciate ways in which specific representations revealed particular aspects of the relationship. For example, teachers often used a table to make arguments about rates of change and linked their symbolic and graphical representations to the context of the task. Additionally, the use of specific mathematical language such as slope and y-intercept as a means to explain function behavior substantially increased.

To characterize teachers' opportunities to learn about creating and connecting representations, we conducted a fine-grained analysis of a conversation in Class 7. The Cal's Dinner Card Deals task in Class 7 featured a graph as a starting representation, and while the task afforded opportunities to generate other representations, it did not call specifically for other representations. As such, it provided an opportunity to measure the extent to which teachers spontaneously made use of function representations. This discussion was identified by fifteen of the twenty-one teachers identified as influencing their learning. The task presented teachers with a graph representing three meal pricing plans and asked them to determine which of plans was best, recording results on a poster. The class then engaged in a gallery walk in which they publicly wrote observations and posed questions about each

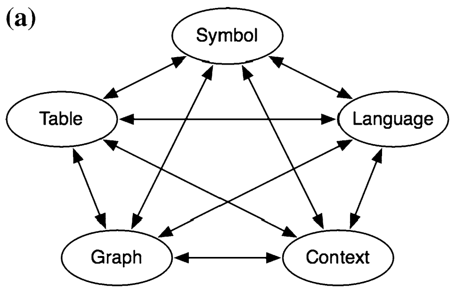

Fig. 4 Five representations of function and the connections made in Class 7

group's solution to the task. The discussion of these solutions, observations, and questions sparked a 30-minute discussion about the task and the underlying functions. This discussion included 336 talk turns and contained speaking turns by the instructor and sixteen of the eighteen teachers present that day.

In our analysis, we coded talk turns for each of the five representations shown in Fig. 4a. For each coded turn, we identified whether a connection was made to another representation. Using arrowheads to indicate connections that were made, Fig. 4b shows the extent to which teachers generated and connected representations in this discussion. (For example, the single-headed arrow between Table and Context indicates that teachers discussed a table and connected mathematical features of the table to the context of the problem situation.) In all, teachers generated four function representations in addition to the initial graph that was presented in the task, and their talk made connections in fourteen of the twenty possible directions in the service of explaining their solutions to the task. In the text that follows, we provide a brief narrative account of the conversation, with excerpts, to illustrate the ways in which teachers made substantive connections between function representations.

The conversation began with observations about similarities and differences between the solution paths. For example, Christopher observed that "none of the [meal values] were continuous… they don't consider half or anything" (context to language), and Marty noted that "Kerry and Diane's graph actually showed the rate of change and y-intercept, so it's easy to look back at their equation and see where both were coming from" (graph to language, graph to equation). This led to a deeper look into the meaning of slope and intersection in the context of the task, in which Melanie connected language and context: "for every meal you purchase, it's six additional dollars," as did Kerry in describing intersection: "That means that for each of those plans, you're getting the same amount of meals for the same price." This conversation blossomed into a discussion of different procedures for solving simultaneous equations, still making use of multiple representations (graph to equation, graph to context, table to context, table to equation, and table to language in the subsequent 25 turns). The conversation culminated in the teachers extending the equations and graphs to find the cost per meal, touching on concepts of limits.

Teachers' written work on the pattern task demonstrated significant changes in their abilities to generate and connect representations. The conversation analyzed in Class 7 represented an opportunity to select and make use of a variety of mathematical representations. The analysis of this conversation and the excerpts above show teachers spontaneously generating and made substantial mathematical use of all five function representations. This stands in contrast to normative practices in mathematics class, where tasks explicitly ask for specific representations in specific orders, with an emphasis on the symbolic as a starting representation and graphs as a destination. As such, teachers' performance on the pattern tasks as well as the connections made in the Class 7 discussion not only shows significant change, but also represents an opportunity to learn that was likely different from their previous mathematical experiences.

Specialized content knowledge: considering the role of definition

Just as teachers need a range of examples and representations of functions to promote rich understandings in their students, they also need to be able to make informed decisions about the definition of function. Aside from knowing the core components the definition, teachers need to be able to compare and select appropriate definitions for classroom use and consider the role of definition in the classroom. Considering the role of definition includes reflecting on the place that a definition holds in the domain and the understandings it encapsulates. The importance of this aspect of SCK became salient as teachers engaged in the work of the course, linking the work done on developing, revising, and comparing definitions of function to work they might do with students in the classroom.

The team had intended to engage teachers in a critical examination of the definition of function; however, discussions about the role of definition in mathematics more generally were not anticipated in our initial development of the course. The role of definition emerged as a topic as the course evolved and in response, the team designed activities asking teachers to compare three textbook definitions of function and discuss of "A function is a mail carrier" (Sand 1996). Because instrument design preceded this decision about course content, no written or interview items addressed the role of definition. However, the majority of the talk turns in these two discussions during Class 7 centered on the role of definition in mathematics and was subsequently revisited at the close of the course when discussing the final class definition of function during Class 11.

To determine whether teachers had the opportunity to learn with respect to definition, we conducted a thematic analysis of the eleven focal conversations with respect to the role of definition. Two themes emerged in the talk: considering the *mathematical* role of definition and considering the *pedagogical* role of definition. Considering the mathematical role of definition encompassed several key aspects of mathematical definitions: the use of previously defined terms and the notions that definitions are dynamic, open to modification, and arbitrary—that is, the affordances of the mathematical idea being defined are dependent on how the idea is defined.[6] For example, Elaine's comment during a discussion of the "Hair Growth" graph makes salient the idea that definitions are arbitrary:

> But, doesn't this also just kind of show, that sometimes, depending on how you define certain things, that's going to depend how you even graph or define a function… Not everything is cut and dried.

Teachers also talked about the pedagogical role of definition. This talk focused on teachers' evaluations of different definitions of function and whether those differing forms would be useful in the work in *their* classrooms, with *their* students. For example, in the following excerpt, teachers discuss the language of functions with respect to their students:

Terry:	So I never talk about output, for instance, for number two [referring to Fig. 3]. I always tell them that the input controls the output, and then it avoids the passive tense, and I don't know, I think it's somehow more precise, for some reason
Instructor:	So, the input controls the output?
Terry:	That's how I do it with my students
Instructor:	OK. So the message we're trying to send is that there is this dependency. What you get out depends on what you put in, and you're talking about it in terms of control

…

[6] Consider, for example, the two popular definitions for a trapezoid. One definition defines trapezoid as having at least one set of parallel sides, affording the use of any result proven for a trapezoid to be applied to parallelograms. Another definition defines trapezoid as having exactly one set of parallel sides, making it a distinct construct from parallelogram. Neither definition is inherently better, but they afford different mathematical opportunities.

Bonnie Just one small comment, to Terry, is I think the reason just basically because of what she said, because in math we say independent variable and dependent variable, saying the output depends on the input probably makes the relationship clearer

This excerpt illustrates the similarities and differences between definition in the mathematical and pedagogical domains. Terry argued for an alternative wording for the definition to use with her students, acknowledging that multiple mathematically equivalent definitions for function exist. Moreover, Terry is concerned about clarity, but with respect to a particular population: her students. Definitions must be clear with respect to the mathematical ideas being used and built upon; however, issues such as passive tense and precision of language are not necessarily a consideration when building a definition in the mathematical community. Yet, these issues are paramount in working with a group of students; if the language used in a definition is incomprehensible to students, learning the concept is unlikely. A definition needs to be accessible to students, yet mathematically accurate so that they will not learn something later that renders the definition false, such as the notion that multiplication makes a number larger (Ball and Bass 2000). While definitions might be *mathematically* equivalent, they are not always *pedagogically* equivalent; this notion exemplifies teacher learning in the course related to the role of definition.

Policies, standards, and guidance about language use in mathematics education often focus on the use of vocabulary and technical and correct language (Morgan 2005). Discussions of the ways in which different language in definitions might afford different mathematical understandings and implicate different pedagogical uses are outside the norm for many mathematics teachers. The analyses of these discussions suggest that teachers had and made use of the opportunity to consider the ways in which the construction and language of a definition can influence their own and their students' mathematical understandings. While we did not initially identify this as an important topic in the planning of the teaching experiment, providing teachers the opportunity to consider the role of definition represented a significant opportunity to learn specialized content knowledge. Moreover, it provided an opportunity to consider the role of definition beyond just functions, laying the groundwork for transfer to other mathematics content areas.

Discussion

The purpose of this study was to describe changes in teachers' common and specialized content knowledge of function following a content-focused methods course. In analyzing learning through written work and talk, we contend that the course provided significant opportunities to develop common and specialized content knowledge related to function. As such, we consider this teaching experiment to have been successful. We now reflect on teachers' learning more generally and identify course design features that may generalize to other contexts.

A critical idea that emerges from this analysis is that even for secondary teachers who have disciplinary majors, there is a need for carefully designed experiences that develop their CCK and SCK. Given the position of function in the secondary curriculum both domestically and internationally, it might be reasonable to presume that every high school student should have the ability to define function and identify examples and non-examples of functions prior to collegiate work (CCSS 2010; Cooney et al. 2011; NCTM 2000). As such, one would expect teachers to exhibit some knowledge of function, particularly related to being able to define function and provide some core examples (such as linear and quadratic functions). Yet teacher knowledge regarding function and the opportunities teachers report to develop and deepen their knowledge are quite variable (Tatto et al. 2012). Data from the start of the course confirm previous findings—that while teachers are usually able to identify or provide examples of functions, they are less likely to give an accurate definition (e.g., Even 1990; Pitts 2003; Stein et al. 1990). While most teachers were able to classify linear relationships as functions, they were less successful in determining whether more complex relationships were functions. Data from the end of the course indicate significant growth in teachers' abilities to provide a definition and a stronger ability to identify more exotic examples and to create non-examples of functions. When asked to reflect on their learning from the course, eleven teachers—more than half of the class—remarked that they learned about the definition of function.

Similarly, one might expect teachers entering the course to be able to generate multiple representations of functions. Making connections between these representations and being able to interpret a function using multiple representations is an important competency for supporting student learning about function. Our data show that teachers entering the course could generate representations but were limited in their ability to connect them. By the end of the course, the ability to fluently connect representations and to mobilize them in the service of making sense of a function situation was significantly stronger. In particular, the leveraging of visual patterns, as demonstrated on the two written assessment items focused on pattern tasks, and the use of language and context, as seen in the Class 7 discussion represented in Fig. 4b, represent departures from the usual heavy dependence on numeric and symbolic representations in secondary classrooms (Janvier 1987; Leinhardt et al. 1990; Leinhardt and Steele 2005). The work of connecting a generalization back to a visual pattern or context is not trivial, and it serves an important role in making mathematical sense of the underlying function. When a generalization is left without connection to the original context, the complexities of the pattern's composition and the connection between the symbolic and visual or contextual representations are often lost, along with the thought process involved in creating the generalization. The connection back can add explanatory power to the underlying mathematical ideas, such as slope as rate of change and y-intercept. By developing representational flexibility, teachers solidified an important component of SCK that can support their own students' thinking about functions.

These data suggest that the course was worthwhile—while we might expect teachers to have strong CCK related to function, many did not at the start of the course. Moreover, it was successful, in that teachers showed evidence of learning related to function. In particular, teachers showed changes in their SCK, an aspect of mathematical knowledge for teaching that is linked to student achievement and for which there are few systematic opportunities for teachers to develop. While the tasks used with teachers in the course are not necessarily new to the work of teacher education, the ways in which the tasks were assembled and the focus on a single mathematical topic are novel features of the content-focused methods course. We close our analysis by addressing two key questions that help to generalize the work of the teaching experiment: what features of the course more generally supported teacher learning, and what aspects of that learning generalize beyond the study of function?

The content-focused methods course: features that support teacher learning

The content-focused methods course was designed as a means for enhancing teachers' common and specialized content knowledge, addressing a well-documented need in teacher professional education. As evidenced by the work and talk of teachers, the course served this purpose. However, the course was not just an experience in learning content; it was designed to support teachers both as mathematics learners and teachers. We designed the course with three key assumptions in mind: that solving tasks would provide opportunities to develop CCK, discussing solutions would provide opportunities to develop SCK, and considering the tasks as learners and teachers would connect CCK and SCK. We now reflect on what teachers learned in the course and use that reflection to identify three features that supported that learning and that can be applied to the design of other teacher education experiences.

Focusing on a key aspect of mathematics content from the curriculum

The content-focused methods course described in this article focused on a single slice of mathematics content—the notion of function. Function is a topic complex enough to sustain weeks, even months of teacher education and has been shown to pose challenges both for students and teachers. It is a mathematical concept that cuts across grade levels and is identified in standards documents as central to the secondary curriculum. As such, an aspect of mathematics content such as function provides ample opportunity for diverse sets of teachers to engage and do meaningful work that is likely to support their classroom practice in multiple ways. In our course, teachers learned about function in a variety of ways, from refining their own personal definition to expanding their repertoire with respect to examples to considering the ways in which their students might struggle with function concepts. By focusing on function, teachers had opportunities to explore function in a much greater depth than a small collection of function activities within a more mathematically diverse course might afford. This focus also supports work in the classroom; by focusing on one topic, teachers experience the sequencing of tasks and topics in ways that build a conceptual understanding, much in the way that they might design a curricular sequence in their own classroom.

The range in activities also supported learning across the diverse set of teachers enrolled in the course. All teachers entered the course with some experiences with function, yet not the same knowledge and experience. Tasks were selected and facilitated in ways that afforded teachers opportunities to elaborate their knowledge in meaningful ways, no matter their initial understanding of function. Teachers were able to move from flawed definitions of the function concept to correct ones, to add additional ways of defining function to their arsenal, and to consider examples of functions and non-functions that were new to them. With respect to SCK, this diversity also illuminated similarities across diverse populations and contexts, such as when teachers from urban and suburban districts, across middle and high school classrooms, could note similar patterns with their students in the learning of function.

Using a guiding inquiry to frame and motivate the work

During the course, teachers were regularly asked to revisit and refine the definition of function. The sustained and specific inquiry into function was important for a number of reasons. All course activities had links to the concept of function; as such, there always existed a common mathematical thread to which teachers could relate a particular activity. The revisiting of this inquiry pressed teachers to generalize from the specific examples that they had experienced and to consider how the examples may have changed their conception of function. Moreover, the repeated consideration of the definition provided opportunities for the course team to expose and understand teachers' initial misconceptions or naive conceptions about function. In turn, the course activities provided teachers with opportunities to refine and elaborate those initial understandings.

Attending to content, pedagogy, and the ways in which pedagogy can support learning content

Approaching mathematical tasks first as learners, sharing solutions, and then analyzing the tasks as teachers were consistently cited by teachers as an important feature of the course. This sequence provided teachers with opportunities to connect CCK and SCK around a particular aspect of function. In order to meaningfully consider the range of examples, representations, and misconceptions related to a particular mathematical idea, one must first understand the mathematical idea itself. Engaging with mathematical tasks as learners first afforded teachers the opportunity to revisit and revise their common content knowledge en route to exploring the task as a teacher—specialized content knowledge. By engaging as a learner first and then as a teacher around the same task, teachers are more likely to notice finer-grained nuances related to teaching and learning around the mathematical idea, helping them to link their common and specialized content knowledge (Steele 2008). This focus on the content as a learner first afforded teachers a new perspective when considering the content as teachers and in analyzing practice (others' and their own) around teaching the content of function.

The use of practice-based artifacts situated the work in both the content and pedagogy realms simultaneously and was an important feature for linking the two. For example, consider the activity on comparing three definitions of function from textbooks in Class 7, which was cited by 33 % of the class as contributing to their learning. The primary goal of this activity was to consider how the key characteristics of univalence and arbitrariness could be represented in different function definitions. This goal could have easily been accomplished without drawing from definitions from mathematics textbooks or even without explicitly noting the source of the definitions. Yet in doing so, teachers considered not only how the key characteristics of function were evident in the definitions, but also what the implications of using the different definitions might be for work with their students. In this way, the practice-based nature of the activity connected previous work developing CCK to issues of SCK useful in the work of teaching, intimately connecting content and pedagogy.

We argue that these three features of the content-focused methods course teaching experiment were instrumental in providing teachers opportunities to develop their common and specialized content knowledge related to function. We do not argue that these are a complete set of design features nor are they necessarily requisite for content learning in a methods course; rather, we put them forth as a starting point for teacher educators interested in supporting teacher learning of CCK and SCK. Finally, it is important to note that the investigation reported here focused on teachers' content knowledge. As a methods course, the research team held a number of pedagogical goals for the course, and teachers reported learning related to pedagogy as well as content. A detailed report of this learning is beyond the scope of this article.

Generalizing from the particular: potential influences beyond knowledge of function

The primary mathematical goals of the course were to enhance teachers' CCK and SCK related to function; the results presented show that these goals were achieved. The research team had intended for the course to influence teachers in two broader ways: shifting teachers' perceptions of themselves as learners of mathematics beyond the study of function and influencing their teaching practice. We offer a few brief reflections on the ways in which the teaching experiment might have generalized beyond the particulars of learning about functions.

Mathematically, the sustained inquiry in which teachers constructed, revisited, and revised their own understandings models the work of mathematicians in the field. This stands in sharp contrast to the typical practices of texts and teachers, as described by Vinner (1991), in which concepts are embodied by definitions, which are positioned as unilateral authorities by which examples are measured and from which all reasoning about a concept takes place. The positioning of function and its definition in the course as an object of inquiry portrays mathematics instead as an evolving, co-constructive, dynamic body of knowledge that represents a human endeavor. Pedagogically, the sustained inquiry into a mathematical idea might serve as a model for the teachers to engage in their own students. A common criticism of mathematics teaching in the United States by students, teachers, and researchers is that instruction often represents mathematical knowledge as discrete blocks of procedures and tasks that are not related one to the next. The positioning of function in the content-focused methods course, and specifically the use of definition as the lens and touchstone, provides an alternative in which the mathematical explorations serve to build a conceptual understanding of a mathematical idea over time, as imagined by Bruce in his post-course interview:

> I never thought… to go from the problem to the table, or from the graph to an equation… I don't think I've ever done a problem where it started that way. And I thought that [there were] some points to be learned in this discussion that happened in the second to last class about slope, and how kids tend to memorize the m part and… can't put any meaning to it because they just learned it, slope is what comes before x. And really, if it gets most easily seen on a graph, like it's the change in y over the change of x, but if you don't really get there, if you graph second, the kids will grab onto that definition of it's just m and never really get the idea. I think kids in mathematics tend to latch onto the first thing that they're taught and then kind of tune out everything else. So personally, next year when I teach this stuff, I'm going to start with a graph and try and introduce slope that way. Then come back and say "Hey, what do you know- it's also the m in front of this x here."

The question of whether these practices will be realized in the classrooms of teachers who participated is still an open one that would require further study, a limitation of the work. However, we argue that by building and linking common and specialized content knowledge related to function, teachers have taken a requisite first step in being able to engage their own students in deep and meaningful study of functions. Moreover, the teachers in the course engaged in practices that served as a model for a conceptual, learner-centered inquiry into a complex mathematical topic. Teachers moved between individual, small-group and whole-class discussions in which key aspects of function were considered, were pressed to discuss and represent the collective knowledge of the group in a variety of ways, and periodically were asked to reflect on their own learning. They engaged with function both as a learner and as a teacher, giving them the opportunity to both enhance their own mathematical knowledge and to consider issues of their own students' learning. By grounding these student-centered pedagogical practices in the context of learning mathematical content related to function, teachers experienced learning at the interface of theory and practice that is likely to impact not only their own understanding of the mathematics of function, but also the understandings they foster with in their own classrooms.

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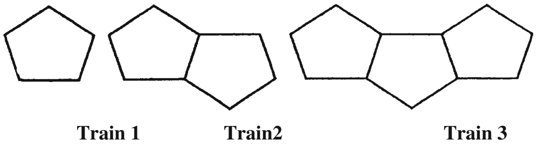
Appendix: Pre-/post-written assessment items

Pre-/Post-test Questions 1–4

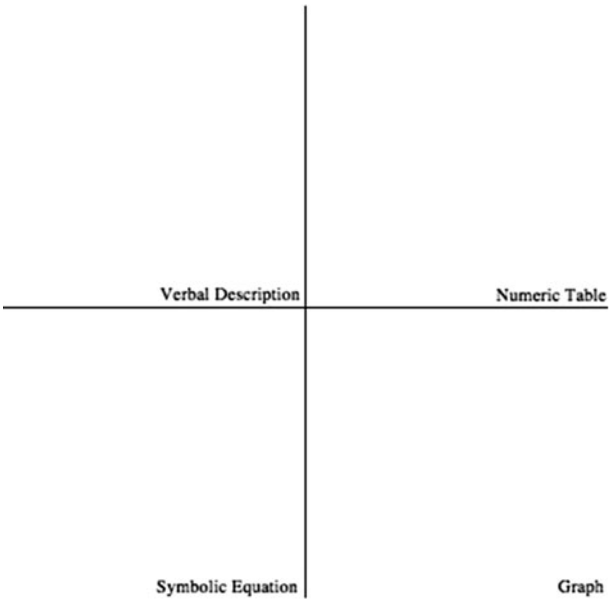
1. The first train in this pattern consists of one regular pentagon. For each subsequent train, one additional pentagon is added. The first three trains in the pattern are shown below.



   a. Determine the perimeter for the 4th train.
   b. Determine the perimeter for the 100th train.
   c. Write a description that could be used to find the perimeter of any train in the pattern.
      Explain how you know.
      How does your description relate to the visual representation of the trains?

2. For each of the situations below, specify the relationship between the quantities using a verbal description, numeric table, symbolic equation, and graph.

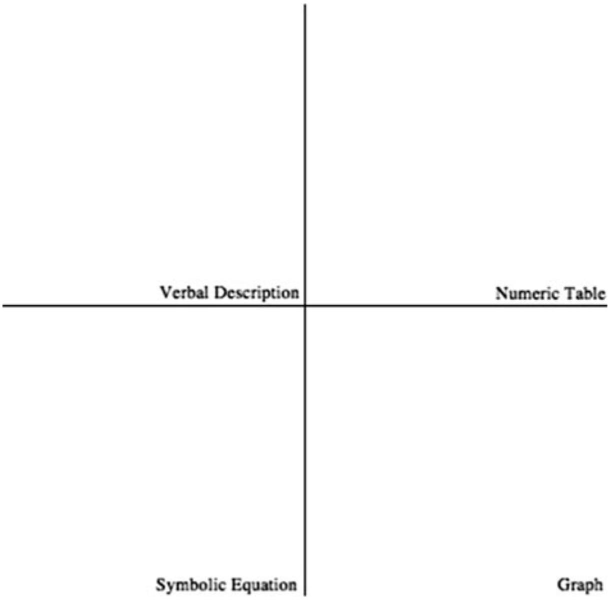
   a. You are buying a number of apples. The apples cost 30 cents each. Specify the relationship between the number of apples and the total cost.



Check off all the terms that apply to this relationship using the list below:

Function	Linear	Proportional
Non-function	Nonlinear	Non-proportional

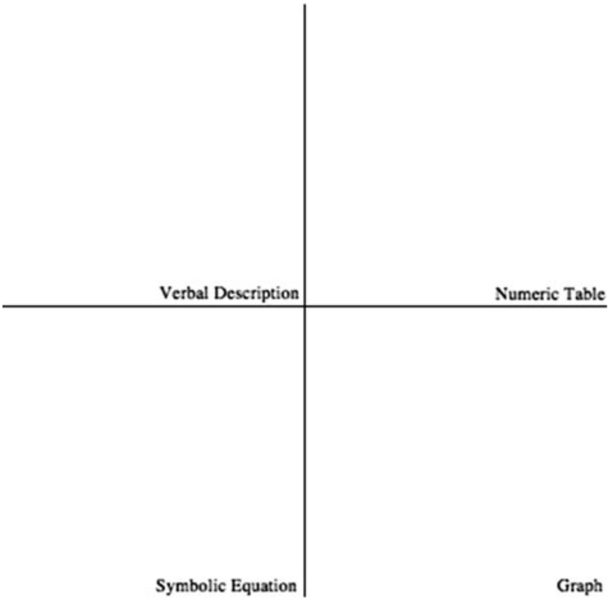
b. A video store charges $25 per month for unlimited rentals. Specify the relationship between the number of videos you rent in a month and the cost per video.



Check off all the terms that apply to this relationship using the list below:

Function	Linear	Proportional
Non-function	Nonlinear	Non-proportional

c. A cable company charges $25 a month for service plus a $50 installation fee. Specify the relationship between the number of months you subscribe and the total amount paid for installation and monthly service.



Check off all the terms that apply to this relationship using the list below:

Function	Linear	Proportional
Non-function	Nonlinear	Non-proportional

3. The figures below are made of squares. The number of squares increases with each new figure.

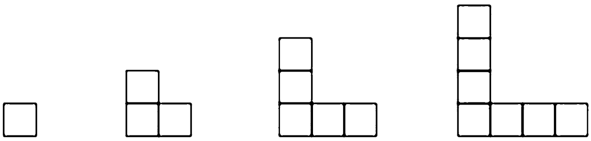


Figure 1 **Figure 2** **Figure 3** **Figure 4**

Imagine that the pattern continues to increase in the same way.

a. How many squares are in the 5th figure?
b. How many squares are in the 100th figure?
c. How could you find the number of squares in any figure in this sequence?
d. Explain how you know.
e. How does your method relate to the visual representation shown above?

4. What is a function? Give an example of a function and a non-function.

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