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A framework for designing a research-based "maths counsellor" teacher programme

Uffe Thomas Jankvist[1] **· Mogens Niss**[2]

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Abstract This article addresses one way in which decades of mathematics education research results can inform practice, by offering a framework for designing and implementing an in-service teacher education programme for upper secondary mathematics teachers in Denmark. The programme aims to educate a "task force" of so-called "maths counsellors", i.e., mathematics teachers whose goal it is to help identify students with genuine learning difficulties in mathematics, investigate the nature of these difficulties, and carry out research-based interventions to assist the students in overcoming them. We present and discuss the various components of the programme, theoretical as well as practical, and account for how these make up a framework for designing a research-based "maths counsellor" teacher programme.

Keywords Students' learning difficulties · In-service teacher education · Professional development of teachers · Relation between research and practice · Mathematical competencies · Design

1 Prologue

We begin this article with an informative labelling, or even a set of "warnings" for the reader. The topic of study addressed in this article is not one of the presently predominant reports, qualitative or quantitative, on empirical mathematics education research. Rather it is to be seen as an example of how to create a framework for putting established mathematics education research results back into mathematics teaching and learning practice. In that sense this study has some resemblance with earlier studies in the field (e.g., Howson, 1975). This also means that the questions which drive this study, and hence this article, are not to be considered as traditional "research questions" but instead as *research-based*, design-oriented developmental

✉ Uffe Thomas Jankvist
utj@edu.au.dk

Mogens Niss
mn@ruc.dk

[1] Aarhus University, Campus Emdrup, Copenhagen, Denmark

[2] Roskilde University, Roskilde, Denmark

questions. In this regard, the article offers and presents (as its main result) a framework for designing and implementing a so-called "maths counsellor" programme for experienced in-service upper secondary level mathematics teachers. Moreover, it should also be emphasized that this programme is not a classical teacher training programme but a state-of-the-art post-graduate course designed for upper secondary mathematics teachers with master's degrees in mathematics and related subjects – and a course with a very specific aim; namely that of turning these teachers into practising "maths counsellors".

2 Introduction

In Denmark, today, seven out of ten 16-year old young people choose to enter into upper secondary education (in a general, a technical, or a business-oriented upper secondary school programme). Only a few decades ago, the percentage of young people choosing upper secondary school was significantly lower, and the increase has happened at the expense of vocational programmes such as apprenticeships in the crafts and trades, in retail, etc. Despite the fact that it was a governmental goal to increase the number of citizens with an upper secondary diploma, the upper secondary schools were not necessarily geared to, nor prepared for, the task of taking on this new population of students, not the least so because the majority of Danish upper secondary school teachers are traditionally trained at university within academic subjects proper, and with very little – sometimes no – subject specific didactics or pedagogy forming part of their studies.

From the point of view of upper secondary mathematics, one of the consequences of the development just outlined is that the number of students with genuine learning difficulties in mathematics has increased.[1] Moreover, quite a few of the upper secondary students have chosen programmes with mathematics as a significant compulsory component without really wanting to study mathematics, simply because mathematics is a necessary prerequisite for entering many tertiary programmes. So, experience and mathematics teachers tell us that in every mathematics classroom there are a number of students who try their best and work their hardest to grasp the mathematics they are supposed to learn, to figure out what is going on, to submit written tasks and to take tests, etc., all with very limited success and without getting anywhere in their pursuit. The number of such students has gone up in all kinds of upper secondary schools as a consequence of the processes outlined. These students experience and display what we term genuine *learning difficulties* that are specific to mathematics, whereas they may not have problems in other subjects except in those in which mathematics plays a marked role. When speaking about genuine learning difficulties *in mathematics* we have in mind those seemingly unsurmountable obstacles and impediments – stumbling blocks – which some students encounter in their attempt to learn the subject. These stumbling blocks include, but are not limited to, a wide range of misconceptions, misinterpretations, misguided procedures, inadequate beliefs etc. with regard to established notions of mathematics. We do *not* include general learning disabilities, cognitive or affective disorders and the like. Moreover, as for the students, we are not concerned with those who lack motivation for investing time and

[1] Danish upper secondary school takes three years, 1–3 g ("g" stands for "gymnasium"), and students usually enter at the age of 16 after having completed ten years of mandatory comprehensive primary and lower secondary schooling. Upper secondary students can choose to have mathematics for one, two or three years; one year being the compulsory minimum level, three years being the advanced level, whereas two years, of course, represent an intermediate level.

effort in learning mathematics. On the contrary, the target group students are those who unsuccessfully try hard to learn mathematics and who are interested in getting help to overcome their difficulties. Hence, the question which initially spurred us to undertake the project under consideration simply is:

- How is it possible to provide research-based assistance to this kind of students so as to help them overcome their learning difficulties in mathematics?

A basic principle for answering this question is that large parts of the assistance should be provided by the students’ own *teachers* in close connection with everyday teaching and learning activities in their school in collaboration with a maths counsellor in the school, not by some “clinical” agency or task force that is external to the school. In other words, the focus of the project is on enabling in-service upper secondary mathematics teachers – and their schools – to provide effective help to students with learning difficulties in mathematics.

We have identified three overarching mathematical domains within which Danish upper secondary students usually encounter particular difficulties, and which we hence deem particularly significant in the Danish context. The first of these is related to *mathematical concepts and concept formation*. The reason for choosing this topic is of course related to the huge body of international mathematics education literature addressing students’ difficulties with concept formation, e.g., the mathematical concepts of number, fraction, equation, function, derivative, and differential equation, just to name a few (cf. the course literature described later in the article). From a national PISA perspective, we know that Danish 15-year old students have difficulties with topics related to algebra, and the use of symbols (Lindenskov & Jankvist, 2013). The second domain is related to mathematical *reasoning, proof and proving*. Students’ difficulties regarding this domain are also well described and analysed in the international mathematics education literature (see below). At the same time, we know from experience that Danish students struggle with accepting and relating to the notion of mathematical proof, also because the topics of mathematical proof and proving have been more or less diluted in Danish upper secondary school during the past couple of decades. As for mathematical reasoning, this is of course not only related to proof and proving, but most certainly also to concept formation, sense-making of mathematical concepts and activities, and not least in relation to the third domain that we have identified, namely that of mathematical *models and modelling*. This third domain has been on the agenda of the Danish upper secondary mathematics programmes for almost 30 years, and with the latest reform of 2005 it was strengthened further, at least in formal terms. Although we treat the entire modelling process we pay special attention to the act of mathematization, since several studies, including the Danish PISA 2012 results, show that this constitutes a major stumbling block for the students. (We shall elaborate more on each of these domains later in the article.) Thus, when dealing with the above-mentioned generating question we have accumulated and selected research results and findings, related to the three domains, presented in mathematics education research literature to constitute our basis for developing an in-service teacher education “maths counsellor” programme. We find that over the past four decades or so mathematics education research has accumulated so massive a fund of knowledge and insights – in the form of a multitude of specific concepts and findings, theoretical constructs, interpretive schemes, efficient teaching and assessment approaches and so on and so forth – related to students’ learning difficulties within these three domains that a research-based maths counsellor programme is warranted. One indicator supporting this view is the series of articles by the EMS Committee of Education, reporting “solid findings” from mathematics education research to a broad audience of mathematicians (Education Committee of the European Mathematical Society, 2011).

We have to specify what we mean by a “maths counsellor” in the present context. A *maths counsellor* is a practising upper secondary mathematics teacher who has undergone targeted in-service education in assisting the kinds of students mentioned above *and their teachers*. More precisely, a maths counsellor, as defined here, is able to (1) identify (detect and select) students with genuine learning difficulties in mathematics, to investigate the nature and origin of these difficulties, and to carry out intervention measures to assist the students in overcoming them, and (2) to work as a peer counsellor with his or her mathematics teacher colleagues so as to “help them help” their students as far as mathematical learning difficulties are concerned.[2]

Against this background, the main question in this article, closely following up on the previous one, is:

- How to design a research-based “maths counsellor” in-service education programme for Danish upper secondary mathematics teachers enabling them to assist students and teacher colleagues to counteract genuine learning problems in mathematics?

It ought to be mentioned that the idea of subject counsellors in schools is not entirely new in Denmark. For quite some time there has been “reading counsellors” assisting students in reading, writing and spelling in upper secondary schools. Also, many primary and lower secondary schools have maths counsellors whose task it is to provide advice to and sparring with their peers on general issues pertaining to the teaching and learning of mathematics. However, the maths counsellors in focus of this article will have quite a different type of background and are meant to have rather different kinds of roles in their schools.

In the following sections of the article, we first put forward a fundamental framework for identifying and analysing students’ learning difficulties in mathematics, namely the Danish so-called KOM project about mathematical competencies. One main reason for bringing in this – multifaceted – framework is that it offers a general perspective from which the three domains of students’ learning difficulties, which we elaborate on later, can be viewed. Also, the KOM framework describes a set of didactic and pedagogical competencies for mathematics teachers. We refer to these competencies throughout this article in relation to in-service teachers’ development towards becoming maths counsellors. We give an account of our most important theoretical and practical design considerations, leading to an outline of the maths counsellor programme, also including further deliberations on the envisioned role of a future maths counsellor at the upper secondary level. Next, we enter into greater detail with aspects of the programme design: e.g., the instruments we have devised for the maths counsellors to use in their work, the literature adopted for the programme, participants’ hand-in (sub)project reports, the final examination, etc. On top of this, we provide three examples of students’ learning difficulties dealt with in the mathematics education literature, which help illustrate the foreseen reality of the maths counsellors’ future work. Finally, we address the two questions driving the

[2] Thus, in this role a maths counsellor is neither a “subject leader” – the first amongst peers – of mathematics teacher colleagues in a school, nor a “mentor teacher” in the classical sense of an experienced teacher offering didactico-pedagogical advice on all matters mathematical to, or supervision of, less experienced teachers. Rather, a maths counsellor is first and foremost a counsellor who is specialized in dealing with mathematics specific learning difficulties, vis-à-vis students or colleagues.

project and this article, and share some preliminary but significant and illustrative experiences from the ongoing implementation of the programme.

3 Mathematical competencies

The Danish KOM framework[3] of 2002 characterises *mathematical competence* as “…having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role” (Niss & Højgaard, 2011, p. 49). This overarching mathematical competence is spanned by eight distinct, yet mutually related, *competencies* – the use of “competence” and “competency” being fully deliberate for the sake of distinction. A mathematical *competency* is (an individual’s) “*...well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge*” (ibid.). The eight competencies are:

- mathematical thinking competency,
- problem handling competency,
- modelling competency,
- reasoning competency.
- representation competency,
- symbols and formalism competency,
- communication competency,
- aids and tools competency.

We shall describe each competency only very briefly. For more detailed accounts, lists of examples, etc., the reader is referred to Niss and Højgaard (2011) (also cf. Kilpatrick, 2014).

The *mathematical thinking competency* comprises an awareness of the types of questions that are typical of mathematics, and an insight into the types of answers that can be expected. Also the acts of abstracting and generalising are part of this competency, as is the act of distinguishing between different categories of mathematical statements such as conditional (“if-then”) claims, definitions, theorems, phenomenological statements about single cases, and conjectures. This competency also includes being able to understand the nature and role of (verbal or symbolic) quantifiers in such statements.

The *problem handling competency* involves the ability to detect, formulate, delimit, and specify different kinds of mathematical problems, pure or applied, as well as the ability to solve mathematical problems that have already been formulated by oneself or by others. An important thing to notice about this competency is that the word “problem” is relative to the person who is trying to solve the task; what to one person is a routine task may be a problem to another, and vice versa.

On the one hand, the *modelling competency* consists of the ability to analyse the foundations and properties of existing models and to assess their range and validity. On the other hand, it involves being able to perform and utilise active modelling, including mathematization, in given extra-mathematical contexts and situations. The boundary between dealing with applied mathematical problems, as in the problem handling competency, and active mathematical model building is somewhat blurred. The more specific the extra-mathematical facts

[3] “KOM” is an acronym for “competencies and mathematics learning” in Danish.

and features of the context, situation and problem taken into account are, the closer we come to model building.

The *reasoning competency* consists, first, of the ability to follow and assess mathematical reasoning, i.e., a chain of arguments put forward – orally or in writing – in support of a claim. In particular it concerns understanding what a mathematical proof is, the basic ideas of a proof, and when a chain of arguments does or does not constitute a proof. For example, this comprises being able to understand the role and logic of a counter example. Secondly, this competency consists of the ability to actually devise, carry out and explain (valid) mathematical proofs.

The *representation competency* comprises the ability to understand, i.e., decode, interpret, and distinguish between, as well as utilise, different representations of mathematical objects, phenomena, problems, or situations (including symbolic, algebraic, visual, geometric, graphic, diagrammatic, tabular, verbal, or material representations). At the same time, the competency includes being able to understand the mutual relationships between different representational forms of the same entity, knowing about their strengths and weaknesses, especially with regard to information content, and being able to choose and switch between representations in given situations.

The *symbols and formalism competency* deals with the ability to decode symbolic and formal language, translate back and forth between mathematical symbolism and natural language, and the ability to handle and utilise mathematical symbolism, including transforming symbolic expressions. Furthermore, it comprises having an insight into the nature of the rules of formal mathematical systems, e.g., axiomatic systems. This competency also focuses on the nature, role and meaning of symbols, as well as on the ways these are used, and on the rules for their usage. Also, the competency is about the handling of formal mathematical systems, whether these are given in a symbolic form or not.

The *communication competency* first consists of the ability to study and interpret others' written, oral or visual mathematical statements, explanations or texts. Secondly, it consists of the ability to express oneself in different ways, registers and genres, and at varying theoretical or technical levels, about mathematical matters, again in writing, orally, or visually.

The final competency, the *aids and tools competency*, first consists of having knowledge of the existence and properties of the diverse sorts of relevant aids and tools employed in mathematics and of having an insight into their capabilities and limitations within different kinds of contexts. Secondly, it comprises the ability to reflectively use such aids and tools, which include not only those related to ICT (i.e., information and communication technology) hardware and software but also, for example, tables, slide rulers, abacuses, rulers, compasses, protractors, logarithmic and normal distribution paper, etc. This competency is about the ability to deal with and relate to all such tools and aids.

4 Didactic and pedagogical competencies

Besides personally possessing the eight mathematical competencies, as well as a variety of general teaching competencies, a good mathematics teacher should also possess a range of specific mathematico-didactic and pedagogical competencies. The KOM framework operates with six such competencies, to be described briefly in the following:

- curriculum competency,

- teaching competency,
- competency of revealing learning,
- assessment competency,
- cooperation competency,
- professional development competency.

The *curriculum competency* consists of, on the one hand, being able to study, analyse and relate to current and future mathematics curricula at a given educational stage, and being able to evaluate the associated plans and the impact on one’s teaching tasks. On the other hand, it also involves being able to draw up and implement different types of curricula and course plans, with different purposes and aims, while taking into account overarching frameworks and terms of reference which may exist under current as well as future conditions.

The *teaching competency* comprises of being able, either alone or in collaboration with students, to devise, plan and carry out concrete mathematics teaching sequences with different purposes and aims. This involves the creation of a rich spectrum of teaching and learning situations for different students and student groups, including the ability to find, judge, select, and produce a variety of means and materials for teaching. It also covers the selection and presentation of tasks and assignments for students. Finally, this competency involves being able to discuss, with the students, the content, forms and perspectives of mathematics teaching, while motivating and inspiring them to engage in mathematical activities.

The *competency of revealing learning* consists of being able to reveal and interpret the actual mathematical learning of students and the extent of their mastery of the eight mathematical competencies, as well as their conceptions, beliefs about and attitudes towards mathematics, including the identification of the development of these over time. Thus, the competency is about getting behind the facade of the ways in which an individual’s mathematics learning and understanding is expressed in concrete situations and contexts, with the intention of grasping and interpreting the cognitive and affective sources of these.

The *assessment competency* comprises of being able to choose or construct a broad spectrum of instruments for revealing and evaluating students’ learning outcomes and competencies, both in relation to specific courses, i.e., to see the extent to which students progress during the course, or in more global - absolute or relative - terms. Also, the competency involves being able to critically relate to the validity and extension of conclusions reached by using given assessment instruments. Finally, the competency involves the ability to characterise an individual student’s learning outcome and mathematical competencies, as well as the ability to communicate with the student about these matters and assist that student to correct, improve, and further develop his or her mathematical competencies.

The *cooperation competency* is, first, about being able to cooperate with colleagues, both in the subject of mathematics and in other subjects, regarding matters relevant to teaching. In this respect, the competency involves the ability to bring the above-mentioned four pedagogical and didactic competencies into play. Furthermore, the competency includes the ability to cooperate with non-colleagues, e.g., students’ parents, administrative agencies, education authorities, etc. about teaching and its boundary conditions.

The *professional development competency* is about being able to develop one’s own competency as a mathematics teacher, in other words a kind of meta-competency. More precisely, it involves being able to enter and relate to activities which can serve the development of one’s mathematical, didactic and pedagogical competencies, taking into consideration changing conditions, circumstances and possibilities. It is about being able to reflect on one’s own teaching and discuss it with mathematics colleagues, being able to identify developmental needs, and being able to select or organise and assess activities – such as in-service courses, conferences or projects – which can promote the desired development. It is also about keeping oneself up-to-date with the latest trends, new materials and new literature in one's field, thus benefiting from research and development contributions, and maybe even about writing articles or books of a mathematical, didactic or pedagogical nature.

5 The target domains of students' learning difficulties

As mentioned in the introduction, at an early stage of this project we identified three target domains related to students' learning difficulties in mathematics, taking into account the Danish upper secondary school context:

1. concepts and concept formation in mathematics;
2. reasoning, proof and proving in mathematics;
3. models and modelling in mathematics.

Hosts of research studies and findings show that concepts and concept formation at large in mathematics constitute major stumbling blocks for students, and that this is the case at all levels of mathematics education (cf. the list of course literature below). Clearly, concepts and concept formation are key prerequisites and components in all the eight competencies in the Danish KOM framework. As to the second and third domains, these are simply instances of two of the eight competencies in this framework. Furthermore, the development and possession of these two competencies rest heavily on other competencies in the framework, all of which in turn depend on concepts and concept formation. In other words, the three domains chosen to be in focus of our project are reflections of key aspects of the eight mathematical competencies and their conceptual underpinnings. As mentioned, mathematical models and modelling are a topic which over the years has gained increasing significance in the Danish upper secondary mathematics programmes, not the least so in the technical programme. For that reason alone it is relevant to focus on this topic here. As previously indicated, the situation is somewhat different when it comes to reasoning, proof, and proving. In the past, proof was a crucial component of upper secondary mathematics in Denmark. It is, however, gradually on its way out, especially when it comes to more rigorous forms of proof. It may therefore seem surprising that we have included this domain in our project. There are two reasons for this. First of all, reasoning does indeed play an essential part in mathematics, not only in pure but also in applied mathematics. This is because reasoning is the way by which mathematical claims of any kind are justified, including solutions to problems. Reasoning certainly encompasses rigorous and formal proof, but goes beyond that. Moreover, proof and proving goes beyond learning textbook proofs of theorems by heart for recitation. Proving deals with the very *act* of justifying claims, statements and solutions. The second reason is that research suggests that many learning difficulties stem from students' experience of absence of sense-making, meaning and coherence in mathematics. Mathematics appears, to many, as a collection of arbitrarily chosen and entirely disconnected concepts, statements, rules, and procedures, devoid of any coherence and suffering from lack of logical clarity (e.g., Fischbein, 1982; Vinner, 2007). Reasoning, proof and proving constitute part of the glue which is needed for sense-making, meaning, coherence, and clarity. They are, therefore, crucial to both the avoidance and the remedying of learning difficulties.

In designing the maths counsellor programme, the three target domains of learning difficulties were chosen as three themes carrying the structure of the programme. The domains qualify as themes in the sense that they cut across mathematical content areas (topics). This we considered a strength in the design, especially because a given mathematical topic may well enter or be left out of the curriculum at any time in the future (for example, for the time being probability theory is not part of the Danish upper secondary mathematics curriculum, but was so 10 years ago). So, concentrating the attention on overarching mathematical themes rather than on particular topics makes the maths counsellor programme less vulnerable to the curricular changes of the day, and contributes to strengthening the maths counsellors' curriculum competency.

6 Designing the maths counsellor programme

The development phase of the programme was carried out during 2012 and early 2013 at Roskilde University, Denmark, which is also responsible for offering and running the programme. The development of the programme was funded under a project supported by the European Social Fund,[4] and a pilot run of the programme began in September of 2012 resulting in 12 maths counsellors receiving their diplomas in January 2014. Since then approximately 24 in-service teachers have been admitted into the programme every year, i.e., in 2013, 2014, etc. In the following, we present further elements of our design considerations regarding the programme and its implementation.

6.1 Striking a delicate balance between theory and practice

As mentioned in the introduction, our intention was to base the programme on mathematics education research results and findings that we believed could help prepare the education of maths counsellors and inform their future work, while also strengthening their mathematics curriculum competency and their learning revealing competency. Whilst it was clear that the programme could not be based solely on the teachers' own experiences and intuition – if so, students' learning difficulties would already have been resolved! – it was also clear that we needed to find a way to transform research results, findings, theoretical constructs, etc. into practice in order to make them meaningful, relevant and applicable to the prospective maths counsellors. Hence, a delicate balance had to be struck between research and practice, a balance which has been widely discussed in the literature on professional development of mathematics teachers (see e.g., Kieran, Konrad & Shaughnessy, 2013; Sowder, 2007).

Our approach to striking such a balance was to introduce a practico-empirical dimension into the programme. More precisely, and closely related to the competency of revealing learning, we made it part of the course – and a condition for passing it – that the teachers should identity, observe and work with actual upper secondary students with learning difficulties in mathematics during the course. This would be a way for them to gain authentic experiences with their future job/task as part of becoming prepared for it. In other words, the programme makes use of a combination of "learning by doing" and "putting research into

[4] The project was called the "STAR-project" and was administered by Jesper Kampmann Larsen and Bent C. Jørgensen.

practice." In this process the teachers were to be supervised by the course directors. More specifically, each teacher would be asked to identify and select one to three "candidate students" in classes he or she was currently teaching, to document the difficulties of each student, and try to determine – through interactions with the individual students – the nature and origin of the observed difficulties. Based on this and drawing on their teaching competency, the teachers then were to design some individual counselling sessions or classroom teaching units, with the purpose of assisting the students selected in overcoming their difficulties. Finally, teachers would be asked to try to assess the range and degree of success of these sessions or units – a task involving the assessment competency. It should be emphasised that the students selected for assistance would remain in their mathematics classes and take full part in the usual teaching-learning activities in the classroom. It should also be emphasised that the students selected were invited to participate in the project on a totally voluntary basis. As a matter of fact, it was pointed out to them that they shouldn't accept the invitation unless they were willing to invest effort and time in the undertaking.

6.2 The three phases of maths counselling

The considerations above led us to describe the envisioned role of a maths counsellor in terms of three phases (or components) of maths counselling drawing on a selection of the didactic and pedagogical competencies mentioned above. A maths counsellor should be able to:

1. *identify* (i.e., detect and select) students with genuine learning difficulties in mathematics;
2. *diagnose* the learning difficulties of the student(s) identified; and
3. undertake *intervention* according to the diagnosis arrived at with respect to the individual student.

In the *identification phase*, the maths counsellor will first draw on a set of research-informed, specially designed instruments – named "detection tests" – the purpose of which is to detect students with mathematics-specific learning difficulties pertaining to the theme at issue. Combined with the counsellor's existing knowledge of the students these tests contribute to subsequently selecting the students who should be invited to receive assistance. (A more thorough description of these tests will be given in forthcoming publications.)

Once the counsellor has identified one or a few students to be part of the project, the *diagnostic phase* begins. In this phase the task of the maths counsellor is to find out what the difficulties of the identified students actually consist in, and from where they might stem. As a simple example, say that a student shows difficulties in solving the equation $3x - {}^{1}/{}_{2}x - 17.5 = \sqrt{2}$. The maths counsellor will now have to find out if the student's problem is one of insufficient symbols and formalism competency, e.g., with regard to symbolic manipulation, if it is to do with understanding and working with fractions, decimal numbers, negative numbers, or irrational numbers, with the notion of variable or unknown, with the meaning of the equals sign, with the very notion of equation or solution to an equation, or with some combination of these. Hence, the diagnostic phase, in this case, could include a session where the student is interviewed about his or her attempt to solve the given equation and is presented with an array of related, probably simpler, equations representing variation over the above-mentioned topics, eventually making a diagnosis possible concerning the key to the student's problems with such equations.

Whilst both the identification and the diagnostic phases rely heavily on the maths counsellor’s competency of revealing learning, the *intervention phase* draws on the competencies of teaching and assessment. More precisely, based on the diagnosis obtained, the intervention phase will have to be designed so as to offer the student some kinds of activities which aim at remedying or reducing his or her difficulties in mathematics. Exactly how such interventions are to be designed and implemented, should be left to the maths counsellors to decide. Needless to say, the prospective counsellors have to be exposed to a variety of general approaches for them to consider, choose from and specify in concrete contexts. However, in the design of the programme, we were cautious not to predefine a set of specific interventions. The reasons for this are the following. First, if specific interventions were predefined, we would also have tended to predefine potential difficulties to be remedied by these interventions. A possible danger in this could be that the prospective maths counsellors might focus too narrowly on the predefined difficulties and their associated interventions, and thus perhaps miss out on other or more deeply rooted learning difficulties with the individual student. Secondly, various practical issues may come into play when attempting to implement a specific set of interventions under rather varying circumstances within different schools. For example, one counsellor may have the opportunity to implement a classroom based intervention designed on the basis of a single student’s diagnosis, for instance if the counsellor deems the difficulty identified common enough for the majority of a class to benefit from the intervention, whereas such an intervention may not be a realistic option for another counsellor in a different setting. Thirdly, it is a well-known fact that teaching – and hence also counselling – works better if the teacher possesses enthusiasm. This is more likely to be the case if the maths counsellor implements interventions which he or she has been involved in conceiving and designing.

Although we have, for analytical reasons, described the three phases of maths counselling as being totally separate, this will not necessarily be the case in practice. For example, already in the process of identifying students with genuine learning difficulties early elements of a potential diagnosis are likely to manifest themselves. Similarly, minor elements of intervention may already occur in the diagnostic phase. We may think of the phases as constituting a discretised continuum with possible overlaps. Moreover, it is clear that when assisting a given student the need for revisiting earlier phases, perhaps repeatedly, may well occur. This will be the case, for example, if new difficulties suddenly become revealed in the process of conducting a diagnosis or undertaking an intervention. A re-analysis of certain detection test items or re-diagnosing may then be needed.

6.3 Practical considerations

Practical and local circumstances are significant determinants for undertaking the job as a maths counsellor. To the extent possible, we wanted to take this into account when designing the programme and enrolling participants into it. For example, since the maths counsellors having completed the programme will be the first of their kind in the Danish upper secondary schools, we recommended that each of the participating schools, if possible, enrolled two teachers instead of just one into the programme. As a maths counsellor, it will undoubtedly be easier to set up and carry out the maths counselling function in a school together with a companion.

Since students’ difficulties in mathematics may not only manifest themselves in mathematics classes, but also in other mathematics-laden subjects such as the sciences and the social sciences, an important element of functioning as a maths counsellor involves collaboration with colleagues in other subjects, requiring a well-developed cooperation competency on the maths counsellors' part. However, we never envisioned the maths counsellors to be supervisors of their teacher colleagues in mathematics or in other subjects, but peers who can offer well-founded sparring.

7 The programme

We now turn to describing the actual programme as it has been implemented since September 2012 and onwards. In order to align the structure of the programme with the three overarching themes, it was decided to let it run over three semesters. So, the first implementation ended in January 2014. Each semester of the programme was initially allotted 5 ECTS points (ECTS: European Credit Transfer System), i.e., the programme counts 15 ECTS in total. This translates to approximately 400 h of further education for the in-service teachers (including preparation, homework assignments, practical work, etc.). Along the road, this allotment turned out to be much too stingy. The actual work load comes closer to 30 ECTS, which was implemented for the third cohort, starting in 2014. In this section, we describe how the teachers, during these three semesters: are exposed to relevant research and other literature; are instructed to use detection instruments developed by us; receive guidance in how to further assist those students (as well as their teachers) who have been identified as having learning difficulties in mathematics; and, finally, how they are supervised – by us – to document and report their work and findings.

7.1 Three semesters

Each semester consists of an introductory and a concluding residential course, each of 2 days' duration. The reason for this is that it may be easier for the teachers to go away from their schools for four whole days in a semester instead of in a fixed number of hours per week. Also, this scheme allows for teachers from any part of the country to participate in the programme in a manageable fashion. In the introductory courses the teachers are introduced to the theme of the semester, the designated literature, the detection test related to the theme and to the work to be done in between the residential courses (all of which will be explained in detail below).

The centrepiece of the work in a semester takes place between the two residential courses, when the teachers in small groups of two-three carry out the three phases of maths counselling at their respective schools. That is, they identify and diagnose students with mathematical learning difficulties and intervene accordingly. Each small group will receive face-to-face supervision by us at least twice during the semester. As mentioned, the teachers are asked to document and report their work, findings and analyses. At the end of each semester they hand in so-called (sub)project reports, in which they analyse the data collected by using the literature and applying theoretical constructs to generate findings. An important feature of these (sub)projects is that they are *problem oriented*, i.e., the participants must, as part of their project, pose and investigate a specific problem related to their students' learning difficulties in mathematics within the theme of the semester (to be exemplified later).

At the concluding residential courses, the teachers present their (sub)projects, including their various findings, to each other and take turns acting as reactors to each other's projects. This idea is taken from the bachelor's programmes at Roskilde University, where every student group has a reaction group and acts as one itself. Experience shows that by having to critically evaluate another group's report (and work), the students learn a lot about their own project work and report writing. For the maths counsellors this is also a way of introducing a culture of constructive criticism, competent feedback, and professional discussion – i.e., of furthering their professional development competency.

7.2 Three lists of literature

In each semester the teachers are provided with a list of literature, divided into three categories:

- General literature
- Theme specific literature
- Supplementary literature

The *general literature* contains a selection of relevant overarching theoretical mathematics education literature focusing on conceptual and theoretical constructs, frameworks and solid findings. (The KOM presentation of mathematical competencies of course makes up an overall framework for all three semesters.) The *theme specific literature* deals with relevant issues, concepts, studies and findings related to the theme of the semester. And the *supplementary literature* list offers various additional texts related to the theme, e.g., more basic introductions, review and survey articles, suggested anthologies, handbooks, etc., but also meta-literature on the role of frameworks, how to do research in mathematics education, how to report findings, etc. Our criterion for the selection of the general and the theme specific literature was that it should foster understanding of the mathematical learning problems which arise in the teachers' everyday practice. In the following, we shall give examples of some of the general and specific literature used in each of the three semesters. Also a few examples of supplementary literature are included.

The first semester general literature consisted in introductions to classical constructs from mathematics education research, which may be used to identify and articulate students' difficulties with mathematical concepts and concept formation and with mathematical understanding. Examples of the constructs used are: Mellin-Olsen's and Skemp's (slightly differing) notions of instrumental, relational and logical understanding (Skemp, 1976, 1979); the process-object duality proposed by Douady (1991) and Sfard (1991); Vinner's and others' distinction between concept definition and concept image (e.g., Tall & Vinner, 1981; Vinner & Dreyfus, 1989); and Duval's (2006) notion of multifunctional and monofunctional semiotic registers. The specific literature addressed topics from mathematics pertinent to upper secondary school, but also more basic ones. Examples are students' difficulties with the concept of function (e.g., Tall, 1992), the concept of limit (e.g., Cornu, 1991; Juter, 2005), and the concept of derivative (e.g., Zandieh, 2000), all topics which show up in a more or less articulate form in the mathematics programmes at upper secondary school. However, knowing that the type of students we were aiming to help often encounter difficulties related to symbols and formalism as well as to representations, the specific literature also included texts on algebra, in particular equation solving, (e.g., Davis, 1975; Filloy & Rojano, 1989; Vlassis, 2002). If students have difficulties with simple algebraic equations, one of the potential roots of these difficulties is an underdeveloped concept of number. So, the specific literature also encompassed studies on students' number concepts (e.g., Fischbein, Jehiam & Cohen, 1995; Markovich & Sowder, 1991; Sirotic & Zazkis, 2007).

The general literature for the second semester covered constructs such as Brousseau's (1997) didactical contract; the notion of sociomathematical norms (Yackel & Cobb, 1996); mathematical thinking and problem solving as approached by Schoenfeld (1992); and students' mathematics-related beliefs (e.g., Op't Eynde, de Corte & Verschaffel, 2002).[5] Although of general interest, these constructs are all in some particular way relevant to the theme of the second semester as well, i.e., reasoning, proof and proving. The specific literature of course deals more explicitly with the elements of the theme. The prospective math counsellors read texts on the different roles and functions of proof (e.g., de Villiers, 1990; Dreyfus & Hadas, 1996; Hanna, 1990; and Niss, 2005), students' difficulties with mathematical reasoning and proof (e.g., Bell, 1976; Dreyfus, 1999; Healy & Hoyles, 2000; Hoyles & Küchemann, 2002), potential means for intervention regarding proof and proving processes (e.g., Knipping, 2008; Leron, 1985), and not least frameworks for addressing students' approaches to reasoning and proof (e.g., Epp, 1998; Harel & Sowder, 2007). In particular, we – and the participants – found Harel's and Sowder's notion of proof schemes useful for the maths counsellors in the diagnostic phase.

The third semester's general literature included a more dynamic approach to growth in students' understanding (Pirie & Kieren, 1994) than what the participants had been presented with during the previous semesters; an encounter with Vygotsky's (1978) zone of proximal development; the first chapter of Freudenthal's (1991) *China Lectures*, which also discusses Treffer's (1987) notion of horizontal and vertical mathematization; and finally an up-to-date and neutral discussion of the use of technology in mathematics education (Trouche et al., 2013), which we deemed relevant in terms of interventions related to students' difficulties with models and modelling. The specific literature[6] for that semester dealt with competencies and modelling (e.g., Blomhøj & Jensen, 2003; Maass, 2006); the modelling cycle (e.g., Ferri, 2006; Niss, 2010a); modelling versus problem solving (e.g., Højgaard, 2010; Lesh & Doerr, 2003); and of course students' prerequisites and difficulties related to the modelling process (e.g., Galbraith & Stillman, 2006; Gravemeijer, 2007). As a general introduction to the theme of the semester, the teachers were given the introduction chapter by Niss, Blum & Galbraith (2007) from the ICMI-Study on mathematical applications and modelling (Blum, Galbraith, Henn & Niss, 2007).

The supplementary literature consisted of the two NCTM handbooks (Grouws, 1992; Lester, 2007) for all three semesters as proposed works of reference; Tall's (1991) *Advanced Mathematical Thinking* for the first semester; the ICMI-Study on proof and proving (Hanna & de Villiers, 2012) and Leder, Pehkonen & Törner's (2002) anthology on mathematics-related beliefs, for the second semester; and for the third semester the above-mentioned ICMI-Study 14 (Blum et al., 2007) and the book resulting from ICTMA 13 (Lesh, Galbraith, Haines & Hurford, 2010).

As the reader will have noticed, the literature of the three semesters, apart from the supplementary literature, contains only a small number of handbook chapters and other introductory texts. This is not a coincidence. One of our "philosophies" when choosing literature was to go to the original and classical texts. One reason for this was our wish to illustrate to the teachers how *research* is actually carried out in mathematics education, what

[5] In that semester the participants benefitted from also reading the three EMS "solid findings" reports on didactical contract; beliefs; empirical proof schemes; and sociomathematical norms.

[6] Again, the prospective counsellors benefitted from the EMS "solid findings" report on models and modeling in mathematics education as an introduction to the theme of the semester.

constitutes a *finding* in mathematics education research, how *data* is collected, analysed and interpreted using theoretical constructs, etc., since they will be using similar techniques in their future job function as maths counsellors.

Finally, it deserves to be mentioned that in several cases the participants had to find (or to be assisted in finding) and utilise supplementary research literature elucidating specific points and issues emerging from the specific investigations they conducted during the semester. Clearly this aspect of the programme contributes to the maths counsellors' professional development competency.

7.3 Three detection tests

Drawing on the research findings available (including the literature mentioned above), one of our first tasks as designers of the programme was to devise the previously mentioned instruments for detecting and selecting students with substantive problems, with regard to their understanding and degree of possession of mathematical competencies concerning those aspects of mathematics which are fundamental for "making it" in upper secondary school in Denmark. More precisely, three "detection tests" were devised, one for each of the semester themes. The tests were presented to the upper secondary students under the title of "*N Questions from the Professor*" (a word play on a standard idiom in Danish, "20 questions *for* the professor"). The number of questions, *N*, in the tests vary from 57 in the test on mathematical concepts, conventions, symbolism, etc., over 23 on mathematical reasoning, proof and proving, through to 13 on models and modelling. In themselves, these numbers are not so telling, since questions sometimes contain sub-questions, but they do illustrate the differences in nature amongst the three themes, based on how many questions a typical upper secondary student would normally be able to answer in approximately 90 min without feeling under any time pressure (a typical double-session in Danish upper secondary school takes up 90 min).

We shall not enter into the specific questions of the three detection tests here, since a proper treatment will require several independent articles (in preparation). Instead, we provide, in a later section, three literature-based examples of students' learning difficulties, which are illustrative of the kind of considerations that guided us in designing the tests. However, it should be mentioned that the tests were devised in such a manner that cross referencing within one test as well as between tests to some extent is possible. For example, if a student in the second detection test shows difficulties in following a certain chain of reasoning, a cross reference may be made to the first detection test regarding the mathematical concepts involved in the argument, in order to find out whether it is the specific mathematical content of the argument that constitutes a stumbling block, or something else, e.g., the logical structure inherent in the argument.

For the first run of the programme (September 2012 - January 2014), the participant teachers were asked to "test the tests", in the sense that they were to pre-identify a small number of students – either in their own classes or in those of their colleagues – who were anticipated to show signs of severe difficulties in the respective tests. For the most cases, the teachers' anticipations turned out to be correct for those students they had pre-identified (they were also found by the test), but more often than not the tests would also single out students whom the teachers initially had believed to be doing reasonably well. One observation regarding the design of the detection tests, the first one in particular, was that from time to time it was difficult for the teachers to distinguish the weakest of the students from those who are “just” weak. This led us to add a handful of extra questions focusing on the weakest students in the revised detection test for the second run of the programme.

7.4 Three (sub)projects

Small groups of two or three participants, preferably from the same school, if possible, carry out the three (sub)projects together, as a group, each in accordance with the theme of the semester. As mentioned, every (sub)project report is centred on a problem oriented question pertaining to the theme, which the prospective counsellors seek to answer by combining empirical investigations and theoretical analyses based on the research literature made available to them. These questions can be of different kinds. As an example, one might ask if there are any significant differences in how a first year class of upper secondary students fare on the first detection test as opposed to a third year class – if “yes”, what are the differences and why exactly these differences, and if “no”, how come? To answer the first part of this question, the participants would have to approach it quantitatively to see if there are any significant differences. However, the second part of the question could be answered by qualitative methods and data collection during the math counselling work of phases 2 and 3, i.e., diagnosing and intervening towards identified students with specific difficulties. As an example related to the second semester theme, one question might ask whether a student’s observed difficulties with respect to reasoning and proof can be explained by links between that student’s beliefs about mathematics and his or her proof schemes (in the sense of Harel & Sowder, 2007). Answering such a question would require in-depth qualitative probes into the nature of the difficulties and the beliefs and proof schemes held by the student.

Against this background it is quite clear that the groups’ choice of a problem oriented question has an impact on the methods which they are going to apply to answer it. Furthermore, the questions – and the answers – will also be closely connected to the relevant theoretical constructs in the semester literature. Although the teachers may not be doing real, publishable”research” in mathematics education in their project work, the approaches in their (sub)projects are not that different in nature from usual mathematics education research, which they have encountered in the semester specific literature. And although the teachers may not obtain original research results, they may well have obtained valid “findings”, most of which will be of an existence nature (i.e., “according to our analysis student *S* possesses such and such specific learning difficulties stemming from such and such sources, which could be partly remedied by such and such intervention means”), but nevertheless findings that may be quite interesting, illuminating and sometimes even surprising. As previously touched upon, another commonality with usual research is that the participants are asked to document and report everything from their diagnostic and intervention work in the schools.

During each semester, i.e., in between the introductory and concluding residential courses, the teachers receive supervision by the programme directors. Typically each group has a couple of supervision meetings, where we discuss everything from problem formulations, the detection test results, methodology, use of theory, the writing of the (sub)project report, to transcription of interview or observation data. We may also watch participants’ video recordings so as to assist them in their analyses of these. Some of these meetings take place in participants’ respective schools, which gives us an opportunity to observe the counsellors in their diagnostic or invention work with the students selected. Such observations have proved important, both in relation to assisting the prospective counsellors in diagnosing and helping their students and in supervising them on how to act as maths counsellors (we shall return to this aspect later in this article). Finally, visiting teachers' in their everyday environment and observing their work with their students also allows us to maintain updated knowledge of today's students' mathematical competencies.

7.5 One final report and one final examination

After having completed the third semester, the project groups are to put their three (sub)project reports into one final report which then provides the basis for their final examination. Here, the intention neither is that the groups just staple the three reports together into one and hand them in, nor that they write up an entirely new report based on the previous three ones; but something in between. More precisely, it is expected that the groups at least write an extra "wrapping-up chapter" in which they attempt to reflect on and contrast and compare the three (sub)projects, different as they may be. For example, the groups may focus on the methods employed, the theoretical constructs and frameworks adopted, interrelations of the problems addressed in the three (sub)projects, the role and place of the three phases of counselling in their own practical work, issues concerning the interventions designed, aspects of students' progress as an outcome of these interventions, including the possibility of evaluating this progress in a meaningful manner, and so on and so forth. If a project group has chosen to follow the same students over several semesters, as is sometimes the case, the wrapping-up chapter provides an opportunity for reflection on these students' longitudinal development and the relationship between their learning difficulties and the three target domains. As an input to this last task of the participating teachers, they are presented with some literature dealing explicitly with the reporting of results and findings in mathematics education research, the role of frameworks, etc. (Kilpatrick, 1995; Lester, 2005; Niss, 2010b). It should be added that since 2014 each project group is requested to also write survey and review essays about the publications in the literature lists which are not explicitly employed in their project reports.

The final examination follows the rules of a Roskilde University group examination, in which the group members first take turns in presenting different aspects of their projects. This typically takes up the first half of the examination, whereas the last half is spent on a scholarly interview and discussion phase, in which the supervisors and the external examiner are free to ask questions referring to the report handed in. It all ends with the students receiving a grade from the peculiar Danish seven-grades-scale (−3; 0; 2; 4; 7; 10; 12, where 2 and up are passing grades) as well as a diploma of an upper secondary school maths counsellor.

8 Three illustrative examples of learning difficulties from the literature

Although many mathematical concepts and topics are in focus when it comes to upper secondary students' learning difficulties in mathematics (e.g., the concept of function, the concept of number, the concept of infinity, probability and statistics), we have chosen to discuss one example for each of the three themes. So, the purpose of this section is certainly not to provide a complete overview of students' various possible learning difficulties in mathematics related to the three themes, just to exemplify some documented research findings which we deem significant for the context of Danish upper secondary school, and hence for the future work of the maths counsellors. These examples may also be understood in terms of potential diagnoses for the maths counsellors to make, i.e., once again drawing on their competency of revealing students' learning, as well as providing crucial input for developing their teaching competency.

8.1 An example from solving algebraic equations

Research suggests that students' difficulties in solving algebraic equations are of two rather distinct kinds (see, e.g., Kieran, 2007). The *first kind of difficulty* – the one most often recognized by teachers and others – is to do with goal-oriented transformation of equations (and, more fundamentally, algebraic expressions) into equivalent ones by way of permissible operations. More specifically, two issues are at play here. First, how to choose a sequence of efficient operations leading to fruitful transformations of the given equation into one from which the solution(s) can eventually be directly inferred – a matter related to the problem handling competency. And secondly, what operations are actually permissible with respect to a set of fundamental rules and conventions so as to preserve solution sets from step to step? This is related to the symbols and formalism competency. The legality of these operations is closely related to the nature and structure of the number domains involved. Many students have difficulties in both respects.

The *second kind of difficulty*, which appears to be of a more fundamental nature, is to do with what an equation actually is, and with what is meant by a solution to it. It may seem surprising to observe that quite a few students are able to correctly solve the equation $7x-3=13x+15$, but are unable to subsequently tell whether $x=10$ is a solution to the very same equation (Bodin, 1993). This suggests that to some students the process of solving equations reside in one mental compartment, whereas the conceptual understanding of what an equation and a solution are is either simply absent or resides in a different compartment. In the latter case, this observation is similar to that of Vinner and Dreyfus (1989), who observed that students who are capable of providing a correct (Dirichlet-Bourbaki) concept definition of a function may not be able to invoke this definition to determine whether an object presented to them is in fact a function or not, especially in cases where the function is not given in terms of a single symbolic expression. In other words, the definition of function resides in one mental compartment, whereas function specimens reside in some other compartment. The two compartments do overlap, but they are not identical. Such a compartmentalisation is likely to lead to incoherent or even inconsistent concept formation and behaviour. A supplementary explanation of this might be that, in the terminology of Skemp (inspired by Mellin-Olsen), the students possess instrumental but not relational understanding (Skemp, 1976). Or in the terminology of Sfard (1991), students may associate an equation with a prompt and a *process* to determine the unknown(s), instead of conceiving the equation as an object in itself, a predicate containing an equals sign accompanied by the demand to identify those elements (if any) in a given domain which satisfy the predicate, whereas a solution is one such element from the domain satisfying the predicate. Finally, in some cases aspects of the imbalances or discrepancies identified in students' concept formation with regard to algebraic equations may be analysed also in terms of Brousseau's (1997) didactical contract, cf. the next section.

8.2 An example on proof about natural numbers and divisibility

It is well known that students often have difficulties with mathematical proofs, not only in relation to following the logical structure of proofs and to coming to grips with what constitutes a proof, but also as regards understanding *why* we need proofs in the first place (Niss, 1999). Clearly, all of this is closely related to the mathematical reasoning competency. The example which we here use for illustration stems from studies by Fischbein (1982) and Vinner (2007).[7] Whilst one of Fischbein's points was to illustrate that many students do not understand that once a general mathematical claim is formally proved, looking for special cases to check it is logically absolutely redundant and superfluous, Vinner's concern was to point out that many students consider the formal mathematical proof as an inconsequential ritual. In the following, we present the example as used by Vinner.

On day 1, students (in 10th and 11th grade) in an Algebra class were presented with a proof that $6|(n^3-n)$ for any natural number n: $n^3-n=n(n^2-1)=n(n-1)(n+1)=(n-1)n(n+1)$, using that $a^2-b^2=(a+b)(a-b)$. Hence, we have a product of three consecutive non-negative integers, at least one of which is divisible by 2 and one (not necessarily a different one) by 3. As 2 and 3 are mutually prime the product is divisible by 6.

On day 2, the students were asked to prove that $6|(59^3-59)$. The students were provided with three different answers and asked which one they preferred (Vinner, 2007, pp. 8–9):

1. "I computed 59^3–59 and found that it is equal to 205,320. I divided it by 6 and I got 34,200 (the remainder was 0). Therefore, the number is divisible by 6."
2. "One can write $59^3–59=59(59^2-1)$. But $59^2-1=59^2-1^2=(59+1)(59-1)$ according to a well known formula. Therefore, $59^3-59=(59+1)(59-1)=(59-1)59(59+1)$. We got a number which is a product of three consecutive numbers. One of them is divisible by 2 and one of them is divisible by 3. Therefore, the product is divisible by 2×3, namely, by 6."
3. "Yesterday, we proved that every whole number of the form $n^3 - n$ is divisible by 6. 59^3–59 has this form. Therefore, it is divisible by 6."

With a population of 365 high school students, Vinner found that 35 % preferred answer 2; 14 % preferred answer 1; 43 % preferred answer 3; and 8 % did not have any preference.

Based on this, some significant observations can be made. If one considers the above from the perspective of sense-making, then going through the general proof to establish a particular case, as in answer 2, is in principle redundant. But, as Vinner (2007) points out, for students to whom mathematical proof is considered a ritual, answer 2 may easily become the preferred one as it repeats the ritual of the first day in a special case. A slightly different interpretation can be made in terms of Brousseau's didactical contract. Some of the students who prefer answer 2 may think that the task they were asked to do was to demonstrate that they understood the general proof by transferring its steps to a special case. Students' responses can also be interpreted in terms of what Harel and Sowder (2007, p. 809) call *proof schemes*, i.e., "what constitutes ascertaining and persuading for that person". The students in Vinner's study who prefer answer 1 are likely to possess a so-called *empirical* proof scheme, in which evidence from examples, specific numbers in this case, serves as the primary source of validation of general claims. As also noticed by both Fischbein and Vinner (and many others), students' perception of the role and nature of proof is connected to their overarching beliefs about mathematics. If, for example, students believe that mathematics is an empirical discipline, alongside physics, chemistry, biology, etc. (see, e.g., Jankvist, 2011), then an empirical approach to proof, including an empirical proof scheme, makes perfect sense. Moreover, the

[7] Both first presented in a PME session held in Berkeley, California, 1980.

distinction between *proofs that prove* (verify) and *proofs that explain* (e.g., de Villiers, 1990; Hanna, 1990) might also help in an interpretation of the outcomes; perhaps the students who did not choose answer 3, did not see the general proof as a satisfactory explanation of the reason why 6 is a divisor.

In any case, perhaps the overarching difficulty that students have with mathematical proof is its remoteness from any everyday context; or as Fischbein (1982, p. 17) puts it: "The concept of formal proof is completely outside the main stream of behavior." In non-mathematical activities of our daily lives, empirical proof schemes *do* apply all the time, i.e., the larger the number of positive examples, the stronger the corroboration of the conjecture. And unlike in mathematics, in daily life there are often exceptions to general rules – cf. also the everyday usage "the exception that proves the rule" (by being an *exception*). In summary, as the concept of truth is simply different in the two contexts it may not be so strange that students encounter difficulties when confronted with the concept of formal proof.

8.3 An example on modelling and mathematization

It is well known that it is not enough to have a good mastery of mathematics for successfully engaging in modelling, i.e., constructing mathematical models of extra-mathematical situations. In other words, the modelling competency involves much more than mathematical knowledge and skills (e.g., Maass, 2006), even though it certainly draws on all the mathematical competencies of the KOM framework. Dealing with the different phases of the so-called *modelling cycle* (Blomhøj & Jensen, 2003; Niss, 2010a) involves several significant potential stumbling blocks for students (Galbraith & Stillman, 2006). One of the most crucial phases is to undertake *mathematization*, i.e., the step in which selected extra-mathematical questions, objects, assumptions, situations and phenomena have to be translated into corresponding entities belonging to some mathematical domain. Once the mathematization has been completed, mathematical problem solving (evidently invoking the problem handling competency) takes place so as to provide mathematical answers to the mathematical questions resulting from the mathematization step.

Now, one reason why mathematization is so demanding is that it requires *implemented anticipation* (Niss, 2010a). This means that the modeller has to anticipate, before and during the mathematization process, not only what mathematical entities may possibly serve to represent the extra-mathematical situation at issue, but – even more importantly – how handling these entities by way of mathematical processes may eventually provide answers to the mathematized questions. So, undertaking mathematization requires that this anticipation be implemented during the very process of mathematization, i.e., before it has been carried out. It is no wonder that this paradoxical state of affairs gives rise to learning difficulties with regard to mathematical modelling. In the first instance, implemented anticipation is a theoretical and analytical construct. However, its presence and significance has also been confirmed in empirical studies. Thus, Stillman and Brown (2014) have shown that implemented anticipation is indeed essential for modellers to be able to carry out mathematization. More specifically, by studying cases of 9th, 10th and 11th grade students working on different kinds of modelling tasks – for instance determining the optimal position of soccer players attempting a shot on goal on a line parallel to the side line – they have found that the ability to implement anticipation of, say, linear relationships between variables is a key driver of successful mathematization, and – more importantly in the context of learning difficulties – that insufficient ability to undertake implemented anticipation is a serious impediment to this endeavour.

For example, when a modelling situation dealing with wave energy calls for mathematization involving certain sorts of functions outside the scope of some students’ knowledge base (e.g., transformed trigonometric functions), or when data points need to be fitted by, say, power or quadratic functions, students tend to choose functions from their repertoire of familiarity (e.g., linear functions) and enforce their application even though the functions’ properties make them inadequate for the purpose and context of the modelling task at issue.

9 Discussion and concluding remarks

As explained in the introduction, the overarching question which has guided us in our work was: how can we effectively assist the kinds of upper secondary students who try hard to learn mathematics but for whom nothing seems to really work. The strategic side of the answer to this question was then to design an in-service further teacher education programme so as to educate a “task force” of maths counsellors, whose job will be to assist such students and their teachers. Our approach to addressing the content side of the question was, first, to zoom in on three specific domains related to upper secondary students’ learning difficulties in mathematics: concepts and concept formation; reasoning, proof and proving; and models and modelling. This both reflects the agenda of Danish upper secondary mathematics education and an abundance of mathematics education literature and previously conducted research within these three areas.

This brings us to the answer(s) to our main question of this article. Obviously, any answer to this question must take the form of an example, in this case an example of an actual design of a maths counsellor programme, supported by the considerations behind it. In our presentation, we have accounted for a delicate balance between research and practice which we have aimed at as one crucial design parameter. Also, we have described the three phases of maths counselling originally envisioned and eventually implemented, in addition to the initial practical considerations. Furthermore, we have provided a detailed account of the final design aspects of the programme, including detection tests, literature lists, (sub)projects, etc. Taken together, these accounts constitute our answer – by way of an existence example – to the question of how to design a “maths counsellor” programme. And along the way we have pointed at the reliance on and potential further development of the prospective maths counsellors’ didactic and pedagogical competencies.

The obvious next question is, of course, whether the design turned out to be effective in relation to its purpose? It is much too early to provide a comprehensive, in-depth answer to this question; nor was this the intention with the present article. Still, we shall offer a few observations and insights taken from the first implementations of the programme, based on our own experiences of supervising the prospective maths counsellors, as well as on some of the participating teachers’ reactions to studying and working within the programme.

One of our own main observations from the first couple of implementations ($N=12+23=35$) stems from the supervision sessions in which the participating teachers were observed while undertaking the diagnostic phase. As mentioned above, the purpose of this phase is to attempt to find the sources and causes of a student’s detected difficulties. What often happened in the beginning, however, when the participants were interviewing students about a task and posing related questions concerning students’ thinking, decisions, choices, etc., was their propensity to turn to teaching the students how to deal with the task. This took place especially if a student who had started off on a right track took a wrong turn. Then in many cases the teacher would quickly step in and say something like: “remember that we usually do it like this” or “are you sure about that – try again” etc. For the purpose of teaching this might well be an advisable approach, but for the purpose of diagnosing a student’s difficulties it is not. For, it may well be the case that a wrong route taken by a student can give rise to valuable insights that pave the way for a diagnosis. So, in such situations the participants had to “unlearn” being teachers and take on the role of a maths counsellor. Although this discursive shift did not come easily to the teachers it did provide them with different perspectives. One of them said: “Changing role to not being a teacher and letting the students explain themselves has provided surprising insights.”

Another observation, from the introductory residential courses, has to do with the mathematics education research literature. At first, it was difficult for some of the teachers to get used to reading such literature. This was partly due to the fact that at the time of their university studies they were mainly exposed to mathematics texts proper, only seldom, if ever, to literature on the didactics of mathematics. Nevertheless, we found that relatively older mathematics education literature – e.g., texts by Bell (1976), Davis (1975), Leron (1985), Skemp (1976, 1979), Tall and Vinner (1981), which, as mentioned, often present singular observations and notions in their originally conceived form without being embedded in larger theoretical frameworks – was often easier for the teachers to follow and digest, and thus served as a way into conceptually and terminologically more complex recent texts.

In relation to the content of the literature, we many a time experienced participants recognizing theoretical constructs from own previous practice. Reading about the didactical contract, one teacher said: “I have experienced this, only at the time I didn’t know what it was.” In relation to the notion of mathematics-related beliefs, at the end of the concluding residential courses, a few participants mentioned that at first they did not really see the point, but now they saw students’ beliefs as being (co-)responsible for several of the difficulties observed. Others found the notion of sociomathematical norms to be quite crucial, and as part of their “wrapping-up chapter” explained how they had become aware of the actual sociomathematical norms in their classrooms, which they now negotiated with their students in the counselling sessions. In so doing they noted how these norms differed from those in the students’ regular mathematics classroom sessions, and considered how to possibly modify the latter. In general, many participants articulated their surprise regarding the usefulness for practice of the mathematics education research they had got to know. One teacher said: “It actually makes sense to use theoretical constructs – they are applicable in analysing individual students’ learning difficulties.”

As a positive consequence of having completed the maths counsellor programme, several new counsellors pointed to the side effects this had actually had on their own teaching. When asked during the concluding residential courses, some reactions were: “I have become much more observant of students’ beliefs and I now pay more attention to what I tell them.” “I now speak more with the students about what mathematics is – and this appears to arouse an interest in the students concerning such questions.” “Our approach to teaching is often ‘example’ and not ‘proof’, which is reflected in students’ perception of the subject.” “Students must be faced with much more difficult tasks than the standard routine ones in order to provoke them to engage in reasoning.” One highly experienced teacher said the following about his personal gains from the programme: “You obtain a quite different perspective on your own teaching. The maths counsellor programme equips us with a conceptual framework and practical experiences, which better enable us to understand the students. It is thought provoking how uncertain even the best of our students’ mathematical knowledge is. The theoretical substance of the programme has helped me to grasp the nature of these problems. It is rewarding.” Another new counsellor described her encounter with the literature, and the perspectives this had provided her with concerning her own practice, as a “mind-expanding experience”.

As to the question of whether the design of the programme worked in relation to our first question, concerning helping a certain category of students with learning difficulties in mathematics, we offer a few examples from participants’ counselling sessions with the students they worked with (a much more comprehensive account will be provided in forthcoming articles). The intervention approaches taken by the teachers were quite varying. One pair of teachers arranged so-called “sandwich sessions”, which consisted in working conversations with the identified students during lunch break several days a week, where the school offered them a sandwich per session. Although these sessions were all of a relatively short duration, the large number of meetings over a semester fully compensated for this. The teachers reported that they took the students participating from failing grades (grades −3 and 0) to passing ones (grades 2 and 4), which is a very long step in Denmark. Other prospective counsellors devised interventions that were classroom based but designed so as to serve the purpose of intervention with regard to diagnoses established for the few focus students. Still other teachers had long hauls of combined interview and working sessions with one or two students who often collaborated on tasks while thinking aloud about their work. One such teacher group reported taking a student from the barely passing grade of 2 to the highest grade 12 in an oral examination with external examiners – which is, however, a rare exception, we have to admit. In general, all participants had positive experiences to report from each semester, for example about students who after having received counselling moved to the front (“active”) end of the classroom on their own initiative and began taking a much more active part in what went on during classroom sessions. Of course the “increased amount of teacher attention” which a student experiences during counselling is very likely to have some positive effect in itself. However, this is not sufficient to account for the nature of the progress participating teachers reported as a result of their interventions. In summary, the combination of increased attention and goal-oriented maths counselling seems to be very powerful indeed. It is important to mention, however, that several prospective maths counsellors expressed frustration of the fact that some of their students made less progress than expected and hoped for. One teacher said that he was simply amazed at “how extremely difficult it is to really capture and map a student’s learning difficulties, and to design interventions accordingly.”

Finally, a few words about the state of affairs: The first 12 maths counsellors from the first (pilot) run graduated in January 2014. The second run of the programme resulted in 23 maths counsellors graduating in January 2015. The third run of the programme began in September 2014. Both the second and third runs were over-booked several months before they began. This shows that there certainly is a market for such a programme. So, relying on the goodwill of the “authorities”, we hope to be able to continue the programme in the future. Denmark has more than 200 upper secondary schools, and if each of them is to have two or more maths counsellors, the programme will have participants at least for the next 10 years.

10 Epilogue

We end this article by coming back to the informative labelling offered in the prologue. As remarked, and as hinted to in the title, the main result reported here is *a framework* *for design* of a maths counsellor programme and a report on the implementation of this framework. In addition to being such a framework for design and implementation, this framework draws heavily on and is informed by another, overarching, framework: the theoretical mathematics education competencies framework, KOM. From a student perspective, the KOM framework provides means for articulating students' learning difficulties through the eight mathematical competencies. From a teacher perspective, the KOM framework offers a way of describing potential development for the prospective maths counsellors through the six didactic and pedagogical competencies. Together, the KOM framework and the maths counsellor design framework offer a successful example of putting decades of mathematics education research results and findings back where they ultimately belong; namely into current mathematics teaching and learning practice, where both students and teachers can profit from them. Needless to say, the KOM framework in itself does not constitute a platform for charting and evaluating the short or long term effects of the maths counsellor programme. We intend to deal with this issue in subsequent publications.

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